College of the Redwoods
Mathematics Department
College Algebra

Final Pretest

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Begin Pretest.

1. Consider the function f defined by

$$f(x) = \begin{cases} -x, & \text{if } x < 0; \\ x, & \text{otherwise.} \end{cases}$$

Which of the following is $f(-3)$?

(a) 4 (b) -2 (c) 3 (d) -3 (e) -1

2. Which of the following functions is even?

(a) $f(x) = x^2 - x$
(b) $f(x) = -x^3 + x$
(c) $f(x) = |x|$
(d) $f(x) = -x$
(e) None of these

3. Which of the following functions is *one-to-one*?

I. $f(x) = x^3$

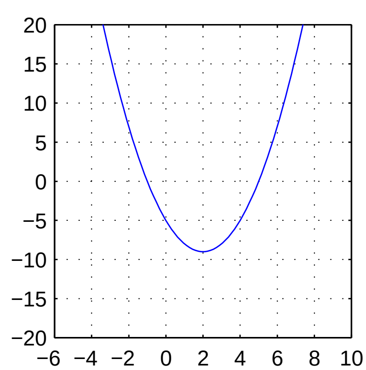
II. $f(x) = x^2$

III. $f(x) = \frac{1}{x}$

IV. $f(x) = x(x+1)(x-1)$

(a) I and II only
(b) I, II, and III
(c) I, II, III, and IV
(d) I and III only
(e) III only

4. Consider the following graph of the function f.



On which of the following intervals is the function f increasing?

(a) $(-\infty, -10)$
(b) $(2, +\infty)$
(c) $(-\infty, 2)$
(d) $(10, +\infty)$
(e) $(-2, +\infty)$

5. Given the equation $f(x) = x^2$, simplify

$$\frac{f(x) - f(2)}{x - 2}.$$

(a) $(x-2)^2$
(b) $x + 2$
(c) $x - 2$
(d) $(x+2)^2$
(e) $\frac{x^3 - 2x^2}{x - 2}$

6. Given the equations $f(x) = x^2$ and $g(x) = 2x + 3$, simplify $f(g(x))$.

(a) $2x^2 + 3$ (b) $2x^2 + 3x$ (c) $2x^3 + 3x^2$
(d) $4x^2 + 9$ (e) $4x^2 + 12x + 9$

7. Which of the following best describes the domain of $f(x) = \log_2(x + 3)$?

(a) $(0, 3)$ (b) $(-3, +\infty)$ (c) $[-3, +\infty)$
(d) $(-\infty, -3)$ (e) $(-\infty, +\infty)$

8. Given that $0 \leq x < 1$, then $|x| + |x - 1|$ equals

(a) $2x - 1$ (b) $2x + 1$ (c) 1
(d) -1 (e) $-2x + 1$

9. Given $f(x) = \ln(x - 3)$, which of the following best describes $f^{-1}(x)$?

(a) $f^{-1}(x) = e^{x-3}$ (b) $f^{-1}(x) = e^{x+3}$
(c) $f^{-1}(x) = e^x - 3$ (d) $f^{-1}(x) = e^x + 3$
(e) $f^{-1}(x) = \dfrac{1}{\ln(x - 3)}$

10. Simplify

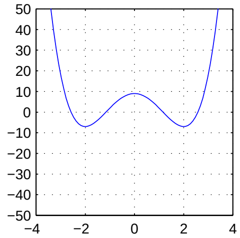
$$\frac{3 + 2i}{3 - 4i}.$$

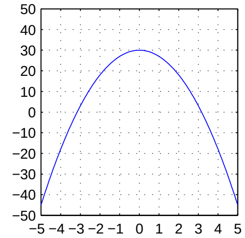
(a) $\dfrac{1}{25} + \dfrac{18}{25}i$ (b) $\dfrac{1}{25} - \dfrac{18}{25}i$
(c) $\dfrac{-3}{25} + \dfrac{13}{25}i$ (d) $\dfrac{7}{25} - \dfrac{14}{25}i$
(e) $\dfrac{11}{25} + \dfrac{19}{25}i$

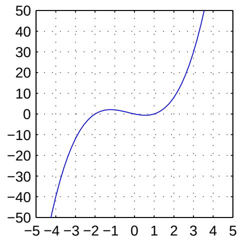
11. Use the quadratic formula to find a complex solution of $x^2 - 2x + 2 = 0$.

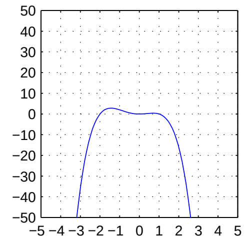
(a) $1 + \sqrt{3}i$ (b) $2 + 2\sqrt{3}i$
(c) $3 - 4i$ (d) $1 - i$
(e) $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$

12. Which of the following polynomials has odd degree?

(a) 

(b) 

(c) 

(d) 

(e) None of these

13. Based on the *Rational Zeros Test*, which of the following could not possibly be a possible rational zero of $f(x) = 2x^3 - 3x^2 - 11x - 8$?

(a) 2 (b) -4 (c) -2 (d) -6
(e) None of these

14. Which of the following is a *vertical asymptote* of the function defined by

$$f(x) = \frac{x^2}{4 - x^2}?$$

(a) $y = 2$ (b) $y = -2$ (c) $x = -1$
(d) $x = 4$ (e) None of these

15. The graph of

$$f(x) = \frac{x^2 - 2x - 2}{4 - x^2}$$

crosses the x-axis at

(a) $2 + \sqrt{3}$ (b) $1 - \sqrt{5}$
(c) $3 - \sqrt{5}$ (d) $1 + \sqrt{2}$
(e) $1 - \sqrt{3}$

16. The rational function defined by

$$f(x) = \frac{x^2}{x - 3}$$

has a *slant asymptote*. What is the equation of this slant asymptote?

(a) $y = x + 3$ (b) $y = x - 3$
(c) $y = x$ (d) $y = 3x$
(e) None of these

17. If \$800 is placed in an account earning 8.25% per year compounded quarterly, how much will be in the account after 5 years? Assume no additional deposits or withdrawals.

(a) \$1054.32 (b) \$1203.41
(c) \$955.56 (d) \$1345.89
(e) None of these

18. How much should be invested at 7% compounded continuously if you need to have \$10,000 in an account at the end of 5 years. Assume that only one deposit is made at the beginning of the 5-year period and that no additional deposits or withdrawals are made.

(a) \$7,046.88
(b) \$5,232.41
(c) \$3,901.12
(d) \$6,755.56
(e) \$12,123.45

19. Estimate $\log_2 9$.

(a) 3.2222
(b) 3.0123
(c) 3.4441
(d) 3.1699
(e) 3.3451

20. The half-life of a radioactive substance is 1000 years. How long will it take 60 milligrams of the substance to decay, leaving 40 milligrams?

(a) 1200 yr
(b) 1322 yr
(c) 960 yr
(d) 585 yr
(e) 700

21. Suppose that a population grows according to the model

$$P = \frac{1000}{20 + 30e^{-0.12t}}.$$

What will be the eventual population after a long period of time?

(a) 50
(b) 1000
(c) 500
(d) 20
(e) None of these

22. Suppose that a population grows according to the model

$$P = \frac{1000}{20 + 30e^{-0.12t}}.$$

What is the initial population?

(a) 20
(b) 200
(c) 2000
(d) 1000
(e) 100

23. Suppose that a population grows according to the model

$$P = \frac{1000}{20 + 30e^{-0.12t}}.$$

How long will it take the population to grow to a level of 40?

(a) 7.6 yr
(b) 8.3 yr
(c) 5.7 yr
(d) 11.2 yr
(e) 14.9 yr

24. If $a > 0$ and $b > 0$, which of the following are identical?

I. $\log a - \log b$

II. $\log \dfrac{a}{b}$

III. $\log(a - b)$

(a) I and II
(b) I and III
(c) II and III
(d) I, II, and III
(e) None of them are equal

25. The expression

$$\frac{1}{2}\ln x - \frac{1}{2}\ln(x+1)$$

is identical to the expression

(a) $\ln\sqrt{x(x+1)}$

(b) $\ln\sqrt{\dfrac{x+1}{x}}$

(c) $\ln\sqrt{\dfrac{x}{x+1}}$

(d) $\ln\frac{1}{2}x(x+1)$

(e) $\ln\big(\frac{1}{2}x - \frac{1}{2}(x+1)\big)$

26. Evaluate the determinant

$$\begin{vmatrix} \lambda & 2 \\ -3 & 4 \end{vmatrix}.$$

(a) $4\lambda - 5$

(b) $4\lambda - 6$

(c) $4\lambda + 6$

(d) $4\lambda + 5$

(e) None of these

27. The inverse of

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

is

(a) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

(b) $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

(c) $\begin{bmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$

(d) $\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}$

(e) $\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

28. If

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \tag{1}$$

then x equals

(a) 1 (b) 3 (c) -2 (d) 2 (e) -5

29. If

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 1 \\ z &= 2, \end{aligned}$$

then x equals

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

30. If

$$\begin{aligned} x + y + z &= 1 \\ x - y + 2z &= 1 \\ 2x - 2y - z &= 2, \end{aligned}$$

use your calculator to determine the value of x.

(a) 1 (b) 3/4 (c) $-1/2$ (d) $-3/2$ (e) 0

31. If

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix},$$

then what is the value of AB?

(a) $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$

32. Use Cramer's rule to solve the system

$$x + 2y = 4$$
$$2x - 3y = 6$$

for y.

(a) 2/5 (b) 3/4 (c) $-5/7$ (d) 2/7 (e) None of these

33. What is the 5th term of the sequence $a_n = \dfrac{n}{n+1}$?

(a) 1/2 (b) 3/4 (c) 4/3 (d) 6/5 (e) 5/6

34. Find the fifth term of the recursive sequence defined by

$$a_{k+1} = 2a_k + 1, \quad a_1 = 1.$$

(a) 15 (b) 31 (c) 55 (d) 63 (e) 88

35. Evaluate the expression

$$\sum_{n=1}^{5} n^2.$$

(a) 45 (b) 55 (c) 65 (d) 75 (e) 85

36. Simplify

$$\frac{n!}{(n-2)!}.$$

(a) $n^2 - n$ (b) $n^2 + n$ (c) $n(n-2)$
(d) $n(n-1)(n-2)$ (e) $n(n+1)$

37. Find the 100th term of the sequence $2, 5, 8, 11, \ldots$.

(a) 293 (b) 303 (c) 296 (d) 302 (e) 299

38. Find the sum of the first 200 odd, positive integers.

(a) 40 (b) 400 (c) $4,000$ (d) $40,000$
(e) $400,000$

39. If $|x| < 1$, then find the sum

$$\sum_{n=1}^{50} x^{n-1}.$$

(a) $\frac{1-x^{49}}{1+2x}$ (b) $\frac{1-x^{50}}{x}$ (c) $\frac{1+x^{49}}{1+x}$
(d) $\frac{1-x^{49}}{1-x}$ (e) $\frac{1-x^{50}}{1-x}$

40. Olive Island has special license plates for its automobiles. The license plates have three uppercase letters of the English alphabet (A through Z), followed by 3 digits (0 through 9). However, the mayor is superstitious and will not allow any license plate to begin with the letter A. How many different license plates can there be if repetitions are allowed? *Note: There are twenty-six letters in the English alphabet.*

(a) 12,812,904 (b) 17,576,000 (c) 16,900,000
(d) 32,000,000 (e) 18,999,000

41. Evaluate ${}_nP_1$.

(a) $n-1$ (b) $n-2$ (c) $n(n-1)$
(d) $(n-1)(n-2)$ (e) n

42. In how many different ways can you arrange 5 people in 3 chairs?

(a) 20 (b) 10 (c) 32 (d) 60 (e) 120

43. In how many ways can you choose 5 marbles out of 7 if the order of choice does not matter?

(a) 42 (b) 25 (c) 21 (d) 14
(e) 2520

44. Expand $(a+b)^4$

(a) $a^4 + b^4$ (b) $a^4 + 4a^2b^2 + b^4$
(c) $a^4 + 4a^3b + 4ab^3 + b^4$ (d) $a^4 + 3a^3b + a^2b^2 + 3ab^3 + b^4$
(e) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

45. What is the third term in the binomial expansion of $(x+2y)^7$?

(a) $32x^5y^2$ (b) $21x^6y$ (c) $84x^6y$
(d) $84x^5y^2$ (e) $21x^2y^5$

Solutions to Pretest

Solution to Question 1: The function f is defined as

$$f(x) = \begin{cases} -x, & \text{if } x < 0; \\ x, & \text{otherwise.} \end{cases}$$

Since $-3 < 0$, select $f(x) = -x$. Thus,

$$\begin{aligned} f(-3) &= -(-3) \\ &= 3 \end{aligned}$$

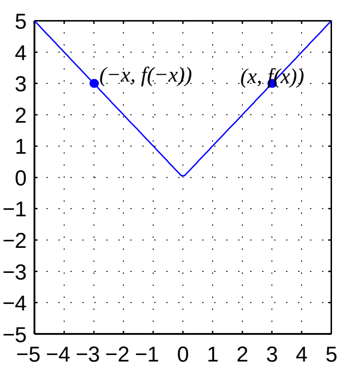
□

Solution to Question 2: If $f(x) = |x|$, then

$$\begin{aligned} f(-x) &= |-x| \\ &= |x| \\ &= f(x). \end{aligned}$$

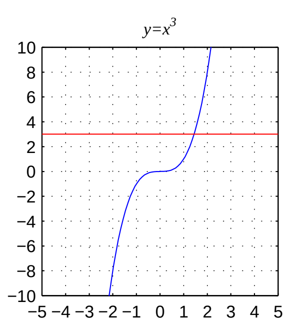
Therefore, f is *even*.

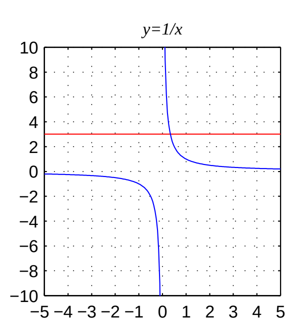
An alternate approach is to consider the graph of the function. An even function is *symmetric with respect to the y-axis*. The graph of $f(x) = |x|$, shown on the next page,



is symmetric with respect to the y-axis and therefore even. None of the other given functions have graphs with this symmetry. □

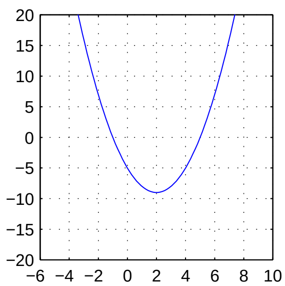
Solution to Question 3: The *horizontal line test* will indicate whether or not a function is one-to-one. If no horizontal line cuts the graph of f more than once, then f is a one-to-one function. If you examine the graphs of $f(x) = x^3$ and $f(x) = 1/x$, then





it is easy to see that no horizontal line cuts either of these graphs more than once. Hence, $f(x) = x^3$ and $f(x) = 1/x$ are one-to-one. □

Solution to Question 4: If you examine the graph of f,



it is easy to see that the graph is *increasing* to the right of $x = 2$. Consequently, the graph of f is increasing on the interval $(2, +\infty)$. *Note: Some might say that the graph of f is increasing on the interval $[2, +\infty)$. This is perfectly acceptable. Unfortunately, this choice is not present in the list of choices, so we must make $(2, +\infty)$ our selection.* □

Solution to Question 5: Recall that the notation $f(2)$ requires us to *substitute* the value 2 for x in the expression $f(x) = x^2$. Consequently,

$$\begin{aligned}\frac{f(x) - f(2)}{x - 2} &= \frac{x^2 - 2^2}{x - 2} \\ &= \frac{x^2 - 4}{x - 2} \\ &= \frac{(x + 2)(x - 2)}{x - 2} \\ &= x + 2\end{aligned}$$

Of course, this argument is valid only if $x \neq 2$. □

Solution to Question 6: This is *composition* of functions. First, evaluate $g(x)$,

$$f(g(x)) = f(2x + 3).$$

Next, evaluate $f(2x + 3)$ by substituting $2x + 3$ for x in the equation $f(x) = x^2$.

$$\begin{aligned}f(g(x)) &= f(2x + 3) \\ &= (2x + 3)^2 \\ &= 4x^2 + 12x + 9\end{aligned}$$

□

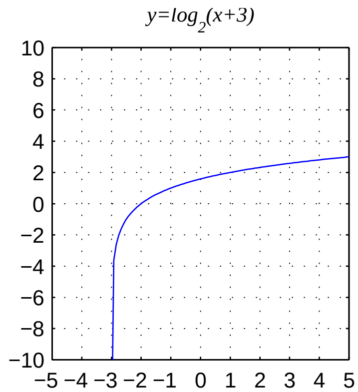
Solution to Question 7: Recall that the *domain* of the logarithmic function is all positive numbers. Consequently, if $f(x) = \log_2(x + 3)$, then $x + 3$ must be *positive*; i.e.,

$$\begin{aligned}x + 3 &> 0 \\ x &> -3\end{aligned}$$

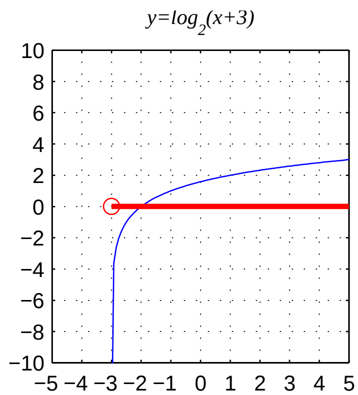
As an alternate solution, one can sketch the graph of $f(x) = \log_2(x+3)$. You can do this on your calculator with the change of base identity,

$$f(x) = \frac{\ln(x + 3)}{\ln 2},$$

or you can simply translate the graph of $\log_2 x$ three units to the left, as shown on the next page.[1]



One can now read the domain of $f(x) = \log_2(x + 3)$ by projecting the points on the graph onto the x-axis, as shown on the next page.



Regardless of your approach, the domain of $f(x) = \log_2(x + 3)$ is $(-3, +\infty)$. □

Solution to Question 8: We analyze the function $f(x) = |x| + |x - 1|$ in *cases*. First, note the following:

$$|x| = \begin{cases} -x, & \text{if } x < 0, \\ x, & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad |x - 1| = \begin{cases} -x + 1, & \text{if } x < 1, \\ x - 1, & \text{if } x \geq 1. \end{cases}$$

Because we are given that $0 \leq x < 1$, $|x| = x$ and $|x - 1| = -x + 1$. Thus,

$$\begin{aligned} |x| + |x - 1| &= x + (-x + 1), \\ &= 1. \end{aligned}$$

□

Solution to Question 9: To find the *inverse* of the function $f(x) = \ln(x - 3)$, one can reason intuitively. The function f first subtracts 3, then takes the *natural logarithm* of the result. Consequently, the inverse must undo each of these operations in *inverse order*. Therefore, f^{-1} must first exponentiate, then add 3. That is,

$$f^{-1}(x) = e^x + 3.$$

[1]Recall that replacing x in $y = \log_2 x$ with $x + 3$ to arrive at $y = \log_2(x + 3)$ *translates* the graph of $y = \log_2 x$ three units to the left.

An alternate approach has one switching the roles of x and y in $y = \ln(x-3)$ and solving for y.

$$\begin{aligned} x &= \ln(y-3) \\ e^x &= y-3 \\ y &= e^x + 3. \end{aligned}$$

Therefore,

$$f^{-1}(x) = e^x + 3.$$

□

Solution to Question 10: To simplify

$$\frac{3+2i}{3-4i},$$

multiply both numerator and denominator by the *complex conjugate* of the denominator.

$$\begin{aligned} \frac{3+2i}{3-4i} &= \frac{3+2i}{3-4i} \cdot \frac{3+4i}{3+4i} \\ &= \frac{9+12i+6i+8i^2}{9+12i-12i-16i^2} \\ &= \frac{1+18i}{25} \end{aligned}$$

Note how we have used the fact that $i^2 = -1$. Now, remember that complex answers must be placed in the form $a + bi$. Consequently,

$$\frac{3+2i}{3-4i} = \frac{1}{25} + \frac{18}{25}i.$$

□

Solution to Question 11: Recall, that if $ax^2 + bx + c = 0$, then the solutions are captured with the *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \tag{2}$$

Thus, if $x^2 - 2x + 2 = 0$, then

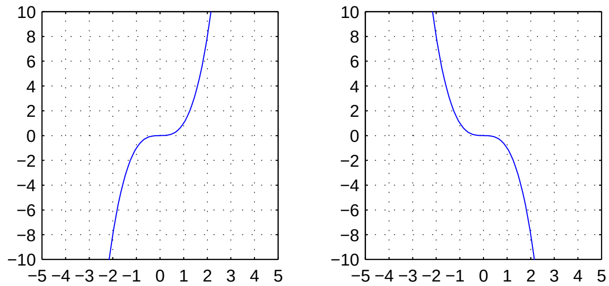
$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}., \\ &= \frac{2 \pm \sqrt{-4}}{2}, \\ &= \frac{2 \pm 2i}{2}, \end{aligned}$$

where we have made use of the fact that $\sqrt{-4} = 2i$. Reducing to lowest terms,

$$x = 1 \pm i.$$

□

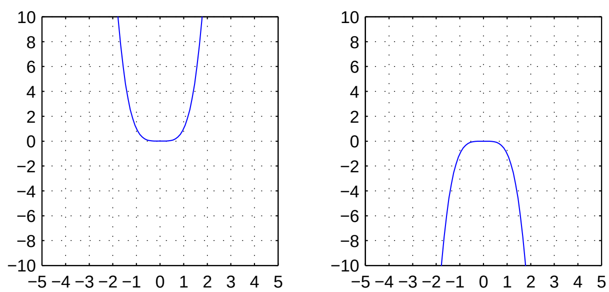
Solution to Question 12: Odd polynomials always exhibit one or the other of the following *end-behaviors*.



That is, odd polynomials either

- fall from positive infinity and continue to fall to negative infinity, or
- rise from negative infinity and continue to rise to positive infinity.

Even polynomials always exhibit one or the other of the following end-behaviors.



That is, even polynomials either

- fall from positive infinity then rise to positive infinity, or
- rise from negative infinity then fall back to negative infinity.

□

Solution to Question 13: The *Rational Zeros Test* tells us that if p/q is a *rational zero* of the polynomial $f(x) = 2x^3 - 3x^2 - 11x - 8$, then p divides the *constant term*, -8, and q divides the *leading coefficient*, 2. Thus, we can list the possibilities for p and q.

$$p = \pm 1, \pm 2, \pm 4, \pm 8$$
$$q = \pm 1, \pm 2$$

The possible rational zeros are formed by taking ratios of the possible values of p and q.

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$$

□

Solution to Question 14: The vertical *asymptotes* of a *rational function* are determined by setting the denominator equal to zero. That is, if

$$f(x) = \frac{x^2}{4 - x^2},$$

then the vertical asymptotes are obtained by setting the denominator equal to zero and solving for x.

$$\begin{aligned} 4 - x^2 &= 0 \\ (2+x)(2-x) &= 0 \\ x &= -2, 2 \end{aligned}$$

Therefore, both $x = -2$ and $x = 2$ are vertical asymptotes of the rational function f. □

Solution to Question 15: The zeros of a *rational function* are obtained by first reducing the rational function to lowest terms, then setting the numerator equal to zero. The rational function

$$f(x) = \frac{x^2 - 2x - 2}{4 - x^2}$$

is already reduced to lowest terms, so we need only set the numerator equal to zero to determine the x-intercepts of the function. We need to use the Quadratic Formula in our calculation.

$$\begin{aligned} x^2 - 2x - 2 &= 0 \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ x &= \frac{2 \pm \sqrt{12}}{2} \\ x &= \frac{2 \pm 2\sqrt{3}}{2} \quad \text{(answer on next page)} \end{aligned}$$

Reducing, we arrive at our final solution.

$$x = 1 \pm \sqrt{3}$$

□

Solution to Question 16: To determine the *slant asymptote* of

$$f(x) = \frac{x^2}{x-3},$$

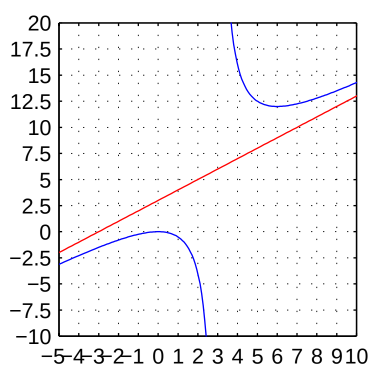
use long division to divide the numerator by the denominator. If you do this, the *quotient* will be $x + 3$ and the *remainder* is 9. Consequently,

$$f(x) = x + 3 + \frac{9}{x^2 - 3}.$$

To find the slant asymptote, let x approach $\pm\infty$.

$$\lim_{x \to \pm\infty} x + 3 + \frac{9}{x^2 - 3} = x + 3$$

Consequently, the rational function f has a slant asymptote with equation $y = x + 3$. In the figure on the next page, the slant asymptote $y = x + 3$ is drawn in red.



□

Solution to Question 17: The formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where

A	Current balance
P	Principal
r	Yearly interest rate
n	Number of compounding periods
t	Time in years

Consequently,

$$\begin{aligned} A &= 800\left(1 + \frac{0.0825}{4}\right)^{(4)(5)} \\ &\approx \$1203.41 \end{aligned}$$

□

Solution to Question 18: The formula for interest compounded *continuously* is

$$A = Pe^{rt},$$

where

A	Current balance
P	Principal
r	Yearly interest rate
t	Time of the loan in years.

Consequently,

$$\begin{aligned} A &= Pe^{rt} \\ 10,000 &= Pe^{(0.07)(5)} \\ P &= \frac{10,000}{e^{(0.07)(5)}} \\ P &\approx \$7046.88 \end{aligned}$$

□

Solution to Question 19: Using the *change of base* rule,

$$\begin{aligned} \log_2 9 &= \frac{\ln 9}{\ln 2} \\ &\approx 3.1699 \end{aligned}$$

□

Solution to Question 20: Because the half-life is 1000 years, half of the initial amount N_0 will decay in the 1000 year period.

$$N = N_0 e^{kt}$$
$$\frac{1}{2}N_0 = N_0 e^{1000k}$$
$$\frac{1}{2} = e^{1000k}$$
$$1000k = \ln\frac{1}{2}$$
$$k = \frac{\ln(1/2)}{1000}$$

Thus, if we start with $N_0 = 60$ mg of the substance,

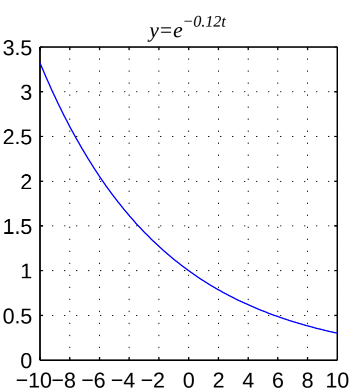
$$N = 60e^{(t\ln(1/2))/1000}$$

To find how long it takes to decay to 40 mg, substitute 40 for N and solve for t.

$$40 = 60e^{(t\ln(1/2))/1000}$$
$$\frac{2}{3} = e^{(t\ln(1/2))/1000}$$
$$\frac{t\ln(1/2)}{1000} = \ln\frac{2}{3}$$
$$t = \frac{1000\ln(2/3)}{\ln(1/2)}$$
$$t \approx 584.96$$

□

Solution to Question 21: If $r < 0$, then the graph of $y = e^{rt}$ *decreases* to zero as time passes.



Consequently,

$$\lim_{t\to\infty}\frac{1000}{20+30e^{-0.12t}}$$
$$= \frac{1000}{20+30(0)}$$
$$= 50$$

Therefore, the eventual population approaches 50. □

Solution to Question 22: If the population grows according to

$$P = \frac{1000}{20 + 30e^{-0.12t}},$$

you can find the initial population by substituting zero for the time.

$$\begin{aligned} P(0) &= \frac{1000}{20 + 30e^{-0.12(0)}} \\ &= \frac{1000}{20 + 30(1)} \\ &= 20 \end{aligned}$$

Therefore, the initial population is 20.

□

Solution to Question 23: Substitute $P = 40$ in

$$P = \frac{1000}{20 + 30e^{-0.12t}}$$

and solve for t.

$$40 = \frac{1000}{20 + 30e^{-0.12t}}$$

$$\begin{aligned} 800 + 1200e^{-0.12t} &= 200 \\ 1200e^{-0.12t} &= 200 \\ e^{-0.12t} &= \frac{1}{6} \\ -0.12t &= \ln(1/6) \\ t &= \frac{\ln(1/6)}{-0.12} \\ t &\approx 14.9\,\text{yr} \end{aligned}$$

□

Solution to Question 24: The logarithm of a quotient is the difference of the logarithms. That is,

$$\log\frac{a}{b} = \log a - \log b.$$

Consequently, I and II are the same. □

Solution to Question 25: Using the laws of logarithms,

$$\begin{aligned} \frac{1}{2}\ln x - \frac{1}{2}\ln(x+1) &= \frac{1}{2}\left[\ln x - \ln(x+1)\right] \\ &= \frac{1}{2}\ln\frac{x}{x+1} \\ &= \ln\left(\frac{x}{x+1}\right)^{1/2} \\ &= \ln\sqrt{\frac{x}{x+1}} \end{aligned}$$

□

Solution to Question 26: Recall that the *determinant* of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad (3)$$

Consequently,

$$\begin{vmatrix} \lambda & 2 \\ -3 & 4 \end{vmatrix} = (\lambda)(4) - (-3)(2)$$
$$= 4\lambda + 6$$

□

Solution to Question 27: To find the inverse of matrix A, set up the augmented matrix

$$[A : I]$$

and reduce.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{bmatrix}$$

This places the augmented matrix in the form

$$\left[I : A^{-1}\right].$$

Consequently,

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}. \quad (4)$$

□

Solution to Question 28: The inverse of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

was found in the Previous Problem. Multiply both sides of the Matrix Equation by the inverse of matrix A.

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore, $x = 3$ and $y = 1$. □

Solution to Question 29: To solve the system

$$x + y + z = 1 \quad (5)$$
$$y + z = 1 \quad (6)$$
$$z = 2, \quad (7)$$

we use *back substitution.* Substitute $z = 2$ in Equation 6.

$$\begin{aligned} y + 2 &= 1 \\ y &= -1 \end{aligned}$$

Next, substitute $z = 2$ and $y = -1$ in Equation 5.

$$\begin{aligned} x + (-1) + 2 &= 1 \\ x &= 0 \end{aligned}$$

Hence, $x = 0$, $y = -1$, and $z = 2$. □

Solution to Question 30: The augmented matrix for system

$$\begin{aligned} x + y + z &= 1 \\ x - y + 2z &= 1 \\ 2x - 2y - z &= 2, \end{aligned}$$

is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 2 \end{bmatrix}$$

Enter this matrix in your calculator to obtain the *reduced row echelon form*

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, the solutions are $x = 1$, $y = 0$, and $z = 0$. □

Solution to Question 31:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (1)(0) + (2)(-1) & (1)(2) + (2)(1) \\ (-1)(0) + (0)(-1) & (-1)(2) + (0)(1) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

□

Solution to Question 32: Do you remember how to calculate a 2×2 determinant? If so, then Cramer's rule states that

$$\begin{aligned} y &= \frac{\begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}} \\ &= \frac{6 - 8}{-3 - 4} \\ &= \frac{-2}{-7} \\ &= \frac{2}{7} \end{aligned}$$

□

Solution to Question 33: Substitute $n = 5$ in the function

$$\begin{aligned} a_n &= \frac{n}{n+1} \\ a_5 &= \frac{5}{5+1} \\ &= \frac{5}{6} \end{aligned}$$

□

Solution to Question 34: We know that

$$a_1 = 1.$$

Substitute $k = 1$, 2, 3, and 4 in the function $a_{k+1} = 2a_k + 1$.

$$\begin{aligned} a_2 &= 2a_1 + 1 = 2(1) + 1 = 3 \\ a_3 &= 2a_2 + 1 = 2(3) + 1 = 7 \\ a_4 &= 2a_3 + 1 = 2(7) + 1 = 15 \\ a_5 &= 2a_4 + 1 = 2(15) + 1 = 31 \end{aligned}$$

Therefore, $a_5 = 31$. □

Solution to Question 35: Start with the lower limit $n = 1$ and end with the upper limit $n = 5$.

$$\begin{aligned} \sum_{n=1}^{5} n^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55 \end{aligned}$$

□

Solution to Question 36: Recall that the *factorial* of a number is defined

$$n! = \begin{cases} n(n-1)(n-2)\cdots 1, & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases} \tag{8}$$

Thus,

$$\begin{aligned} \frac{n!}{(n-2)!} &= \frac{n(n-1)(n-2)(n-3)\cdots 1}{(n-2)(n-3)\cdots 1} \\ &= n(n-1) \\ &= n^2 - n \end{aligned}$$

□

Solution to Question 37: The sequence

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots \tag{9}$$

is called an *arithmetic sequence* with *first term* a and *common difference* d. Note that the n^{th} term of an arithmetic sequence is given by

$$a_n = a + (n-1)d.$$

With these preliminaries out of the way, note that the sequence

$$2, 5, 8, 11, \ldots$$

is arithmetic with first term $a = 2$ and common difference $d = 3$. Consequently, the n^{th} term of the sequence is given by

$$a_n = 2 + (n - 1)(3).$$

Therefore, the 100th term of this sequence is

$$a_{100} = 2 + (100 - 1)(3) = 299$$

□

Solution to Question 38: The sequence of odd integers $1, 3, 5, 7, \cdots$ is arithmetic, with first term $a = 1$ and common difference $d = 2$. Consequently, the n^{th} term of the sequence is given by

$$a_n = 1 + (n - 1)(2) = 2n - 1.$$

Therefore, the sum of the first two hundred odd integers can be written

$$\sum_{k=1}^{200} 2k - 1 = 1 + 3 + 5 + 7 + \cdots 399.$$

The formula for the sum of the first n terms of an *arithmetic series* is given by

$$S = \frac{n(a_1 + a_n)}{2}, \tag{10}$$

where a_1 is the first term of the series, a_n is the n^{th} term of the series, and n is the number of terms in the sequence. Consequently,

$$\begin{aligned} \sum_{k=1}^{200} 2k - 1 &= \frac{200(1 + 399)}{2} \\ &= 40,000 \end{aligned}$$

□

Solution to Question 39: The sequence

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots \tag{11}$$

is called a *geometric sequence* with *first term* a and *common ratio* r. The n^{th} term of the sequence is given by the function

$$a_n = ar^{n-1}.$$

The series

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{12}$$

is called a *finite geometric series*. The sum of this series is calculated with the formula

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1 - r^n)}{1 - r}, \tag{13}$$

where n is the number of terms. Note that the series

$$\sum_{n=1}^{50} x^{n-1} = 1 + x + x^2 + x^3 + \cdots + x^{49}$$

is a finite geometric series with first term $a = 1$, common ratio $r = x$, and there are $n = 50$ terms. Therefore,

$$\begin{aligned}\sum_{n=1}^{50} x^{n-1} &= \frac{1(1-x^{50})}{1-x} \\ &= \frac{1-x^{50}}{1-x}\end{aligned}$$

□

Solution to Question 40: There are 25 choices for the first letter, 26 for the second, 26 for the third, 10 choices for the first digit, 10 choices for the second digit, and 10 choices for the third digit.

The *multiplication principle* tells us that there are

$$25 \times 26 \times 26 \times 10 \times 10 \times 10,$$

or, 16,900,000 possible license plates. □

Solution to Question 41: You might want to review the factorial formula before proceeding. After that, the number of *permutations* or arrangements of n things taken r at a time is given by the formula

$$_nP_r = \frac{n!}{(n-r)!}. \tag{14}$$

Consequently,

$$\begin{aligned}{}_nP_1 &= \frac{n!}{(n-1)!} \\ &= \frac{n(n-1)(n-2)(n-3)\cdots 1}{(n-1)(n-2)(n-3)\cdots 1} \\ &= n\end{aligned}$$

Of course, if you use *common sense*, the number of arrangements of n things taken 1 at a time is clearly n. □

Solution to Question 42: Note that the word *arrangement* is key. This tells us to use the permutation formula. Consequently, the number of ways that we can arrange 5 people in 3 chairs is given by

$$\begin{aligned}{}_5P_3 &= \frac{5!}{(5-3)!} \\ &= \frac{5!}{2!} \\ &= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1} \\ &= 5\cdot 4\cdot 3 \\ &= 60\end{aligned}$$

□

Solution to Question 43: If order doesn't matter, then *combinations* or *choices* are in order here. The number of ways that we can choose r objects out of a set of n objects is given by the formula

$$_nC_r = \frac{n!}{r!(n-r)!}. \tag{15}$$

Consequently, the number of ways that we can choose 5 marbles out of 7 is given by the calculation

$$\begin{aligned} {}_7C_5 &= \frac{7!}{5!(7-5)!} \\ &= \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} \\ &= \frac{7\cdot 6}{2} \\ &= 21 \end{aligned}$$

□

Solution to Question 44: The *binomial theorem* tells us that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Consequently,

$$\begin{aligned} (a+b)^4 &= \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k \\ &= \binom{4}{0} a^4 b^0 + \binom{4}{1} a^3 b^1 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} a^0 b^4 \end{aligned}$$

The *binomial coefficients* are calculated in the same manner as the Choice Formula, it's just a different notation for the same idea. That is,

$$\binom{n}{k} = {}_nC_k.$$

You can also use the fifth row of *Pascal's Triangle* to compute all of the coefficients at once.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

In either case,

$$\begin{aligned} (a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \end{aligned}$$

□

Solution to Question 45: Using the Binomial Theorem,

$$(x+2y)^7 = \sum_{k=0}^{7} \binom{7}{k} x^{7-k} (2y)^k.$$

The first term is generated with $k = 0$, the second with $k = 1$, etc. Consequently, the 3^{rd} term of the expansion is generated with $k = 2$. Therefore,

$$\begin{aligned} 3^{\text{rd}} \text{ term} &= \binom{7}{2} x^{7-2} (2y)^2 \\ &= 21x^5(4y^2) \\ &= 84x^5y^2 \end{aligned}$$

□