College of the Redwoods
Mathematics Department
College Algebra

Chapter 6 Exam—Pretest

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EXERCISE 1. Write out the first five terms of the sequence defined by

$$a_n = 3n + 2.$$

EXERCISE 2. Write out the first five terms of the sequence defined by

$$a_n = -3a_{n-1},$$

given that the first term of the sequence is $a_1 = 2$.

EXERCISE 3. Write out the first five terms of the sequence defined by

$$a_n = a_{n-1} + 2a_{n-2},$$

where the first two terms of the sequence are given by $a_1 = 1$ and $a_2 = -1$.

EXERCISE 4. Find the sum

$$\sum_{k=0}^{5} 2^{k-1}.$$

EXERCISE 5. Consider the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots.$$

Construct a formula for the n^{th} term of this sequence.

EXERCISE 6. Consider the series

$$3 + 6 + 12 + 24 + 48 + 96.$$

Write the series using sigma notation.

EXERCISE 7. Assume that the sequence

$$-5, -2, 1, 4, \dots$$

is *arithmetic*. Write a formula for the n^{th} term of the sequence and use your formula to find the two-hundredth term of the sequence.

EXERCISE 8. Find the sum of the series

$$\sum_{k=1}^{50} 3k + 2.$$

EXERCISE 9. Given an *arithmetic* sequence with $a_2 = 12$ and $a_{20} = 56$, find a_{200}.

EXERCISE 10. Find the sum of the first 100 even integers.

EXERCISE 11. Assume that the sequence

$$-2, -6, -18, \dots$$

is *geometric*. Write a formula for the n^{th} term of this sequence and find the twentieth term.

EXERCISE 12. Given a *geometric* sequence with $a_2 = 12$ and $a_8 = 4$, find a_{20}.

EXERCISE 13. Find the sum of the series

$$\sum_{k=1}^{10} -3(2)^{k-1}.$$

EXERCISE 14. Find the sum of the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots.$$

EXERCISE 15. Find a fraction representation of the repeating decimal

$$0.812121212\dots.$$

Exercise 16. Evaluate

$$_nP_1.$$

Exercise 17. Evaluate

$$_nC_{n-2}.$$

Exercise 18. A typical social security number contains nine digits, such as 012-23-4579. Using this scheme, how many possible social security numbers can be formed?

Exercise 19. On a staff of 60 teachers, three are selected for merit pay for duties above and beyond the call of duty. In how many ways can these three teachers be selected from the pool of available teachers?

Exercise 20. Of 20 teachers in the science division, three will be selected for an ad hoc committee to collect information on new web serving equipment. The first member selected will reside as chair of the committee, the second member selected will be secretary, and the third member selected will handle funds as the committee's treasurer. In how many ways can this three person committee be selected from the available pool of teachers?

Exercise 21. Use the binomial theorem to expand

$$(2x+3y)^5.$$

Exercise 22. Evaluate

$$\sum_{k=0}^{5} \binom{5}{k} 2^{5-k} 3^k.$$

Hint: There is a brute force solution, but there is also a very cute, simple solution.

Exercise 23. What is the third term in the expansion of

$$(a+2b)^{50}$$

Exercise 24. Use mathematical induction to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

Solutions to Exercises

Exercise 1. Substitute $n = 1$, 2, 3, 4, and 5 in the formula $a_n = 3n + 2$.

$$\begin{aligned} a_1 &= 3(1) + 2 = 5 \\ a_2 &= 3(2) + 2 = 8 \\ a_3 &= 3(3) + 2 = 11 \\ a_4 &= 3(4) + 2 = 14 \\ a_5 &= 3(5) + 2 = 17 \end{aligned}$$

Exercise 1

Exercise 2. The first term is

$$a_1 = 2.$$

After that, substitute $n = 2$, 3, 4, and 5 in the relation $a_n = -3a_{n-1}$.

$$\begin{aligned} a_2 &= -3a_{2-1} = -3a_1 = -3(2) = -6 \\ a_3 &= -3a_{3-1} = -3a_2 = -3(-6) = 18 \\ a_4 &= -3a_{4-1} = -3a_3 = -3(18) = -54 \\ a_5 &= -3a_{5-1} = -3a_4 = -3(-54) = 162 \end{aligned}$$

Exercise 2

Exercise 3. Substitute $n = 3$, 4, and 5 in the relation $a_n = a_{n-1} + 2a_{n-2}$.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -1 \\ a_3 &= a_{3-1} + 2a_{3-2} = a_2 + 2a_1 = -1 + 2(1) = 1 \\ a_4 &= a_{4-1} + 2a_{4-2} = a_3 + 2a_2 = 1 + 2(-1) = -1 \\ a_5 &= a_{5-1} + 2a_{5-2} = a_4 + 2a_3 = -1 + 2(1) = 1 \end{aligned}$$

Exercise 3

Exercise 4. Writing out the terms of the series gives

$$\begin{aligned} \sum_{k=0}^{5} 2^{k-1} &= 2^{0-1} + 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} \\ &= 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4 \\ &= \frac{1}{2} + 1 + 2 + 4 + 8 + 16 \\ &= \frac{1}{2} + 31 \\ &= \frac{63}{2}. \end{aligned}$$

Exercise 4

Exercise 5. Write the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

like this:

$$\frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \ldots, \frac{n}{n+1}, \ldots$$

This reveals the n^{th} term.

$$a_n = \frac{n}{n+1}$$

Exercise 5

Exercise 6. In the series

$$3 + 6 + 12 + 24 + 48 + 96,$$

3 is a common factor of all of the terms.

$$3(1) + 3(2) + 3(4) + 3(8) + 3(16) + 3(32)$$

The other factors of the terms are all powers of 2.

$$3(2^0) + 3(2^1) + 3(2^2) + 3(2^3) + 3(2^4) + 3(2^5)$$

Using summation notation yields

$$3(2^0) + 3(2^1) + 3(2^2) + 3(2^3) + 3(2^4) + 3(2^5) = \sum_{k=0}^{5} 3\left(2^k\right).$$

Exercise 6

Exercise 7. The n^{th} term formula for a arithmetic sequence is

$$a_n = a + (n-1)d,$$

where a is the *first term* and d is the *common difference*. The first term of the sequence

$$-5, -2, 1, 4, \ldots$$

is $a = -5$. You can find the common difference by taking the difference of any two consecutive terms. So,

$$d = -2 - (-5) \quad \text{or} \quad d = 1 - (-2) \quad \text{or} \quad \ldots$$

Consequently, $d = 3$ and the n^{th} term is given by

$$a_n = -5 + (n-1)3$$

The two-hundredth term is

$$a_{200} = -5 + (200-1)3 = -5 + (199)3 = 592$$

Exercise 7

Exercise 8. Notice the sum is an *arithmetic* series with common difference $d = 3$.

$$\sum_{k=1}^{50} 3k + 2 = 5 + 8 + 11 + \cdots + 152$$

The formula for the sum of a finite arithmetic series is

$$\sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}, \tag{2}$$

where a_1 is the first term in the series, a_n is the last term in the series, and n is the number of terms in the series. Consequently,

$$\sum_{k=1}^{50} 3k + 2 = \frac{50(5 + 152)}{2} = 25(157) = 3925.$$

Exercise 8

Exercise 9. The n^{th} term of an arithmetic sequence is given by $a_n = a + (n-1)d$, where a is the *first term* of the sequence and d is the *common difference*. Consequently,

$$\begin{aligned} 12 &= a_2 = a + (2-1)d = a + d \\ 56 &= a_{20} = a + (20-1)d = a + 19d \end{aligned}$$

Setup an augmented matrix for this system.

$$\begin{pmatrix} 1 & 1 & 12 \\ 1 & 19 & 56 \end{pmatrix}$$

Place the augmented matrix in *reduced row echelon form*.

$$\begin{pmatrix} 1 & 0 & \frac{86}{9} \\ 0 & 1 & \frac{22}{9} \end{pmatrix}$$

Therefore, $a = 86/9$ and $d = 22/9$ and

$$a_n = \frac{86}{9} + (n-1)\frac{22}{9}.$$

Consequently,

$$a_{200} = \frac{86}{9} + (200-1)\frac{22}{9} = 496.$$

Exercise 9

Exercise 10. The sum of the first 100 even integers,

$$\begin{aligned} \sum_{k=1}^{100} 2k &= 2(1) + 2(2) + 2(3) + \cdots + 2(100) \\ &= 2 + 4 + 6 + \cdots + 200 \end{aligned}$$

is an *arithmetic series*. The formula for the arithmetic series was given in Equation 2. Here $a_1 = 2$ and $a_{100} = 200$ giving

$$\sum_{k=1}^{100} 2k = \frac{100(2+200)}{2} = 50(202) = 10100.$$

Exercise 10

Exercise 11. The n^{th} term formula for a geometric sequence is

$$a_n = ar^{n-1}, \tag{3}$$

where r is the *common ratio* and a is the *first term* of the sequence. The common ratio of

$$-2, -6, -18, \ldots$$

can be found by taking ratios of any two consecutive terms. Consequently,

$$r = \frac{-6}{-2} \quad \text{or} \quad r = \frac{-18}{-6} \quad \text{or} \quad \ldots$$

Therefore, the common ratio is $r = 3$, the first term is -2, and the n^{th} term is given by

$$a_n = -2\left(3^{n-1}\right).$$

The twentieth term is

$$a_{20} = -2\left(3^{20-1}\right) = -2\left(3^{19}\right) = -2,324,522,934$$

Exercise 11

Exercise 12. The n^{th} term of geometric sequence is given in Equation 3. Consequently,

$$12 = a_2 = ar^{2-1} = ar \tag{4}$$
$$4 = a_8 = ar^{8-1} = ar^7 \tag{5}$$

Since this is *not* a system of linear equations, matrices can not be used. Instead use the substitution method. Solve for a in Equation 4.

$$a = \frac{12}{r}$$

Now substitute $a = 12/r$ in Equation 5 and solve for r.

$$\begin{aligned} 4 &= \frac{12}{r}\left(r^7\right) \\ 4 &= 12\left(r^6\right) \\ \frac{1}{3} &= r^6 \\ \left(\frac{1}{3}\right)^{1/6} &= r \end{aligned}$$

Therefore, $a = 12\left(3\right)^{1/6}$ and

$$a_n = 12(3)^{1/6}\left(\left(\frac{1}{3}\right)^{1/6}\right)^{n-1}.$$

Consequently,

$$\begin{aligned} a_{20} &= 12(3)^{1/6}\left(\left(\frac{1}{3}\right)^{1/6}\right)^{20-1} \\ &= 12\left(3\right)^{1/6}\left(\frac{1}{3}\right)^{19/6} \\ &= 12\left(3\right)^{1/6}\left(3\right)^{-19/6} \\ &= 12\left(3\right)^{-18/6} \\ &= \frac{12}{3^3} \\ &= \frac{12}{27} \\ &= \frac{4}{9} \end{aligned}$$

Exercise 12

Exercise 13. The formula for a *geometric* series is

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a\left(1 - r^n\right)}{1 - r}, \tag{6}$$

where a is the *first term*, r is the *common ratio*, and n is the number of terms. So,

$$\begin{aligned}\sum_{k=1}^{10} -3(2)^{k-1} &= -3 - 6 - 12 - \cdots - 3(2^9)\\ &= \frac{-3\left(1-2^{10}\right)}{1-2}\\ &= \frac{-3(1-1024)}{-1}\\ &= 3(-1023)\\ &= -3069\end{aligned}$$

Exercise 13

Exercise 14. Provided $|r| < 1$, the sum of an *infinite geometric series* is given by

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

The infinite series $1 + \frac{1}{3} + \frac{1}{9} + \cdots$ has a first term of $a = 1$ and a common ratio of $r = 1/3$. The series can be summed with

$$\begin{aligned}\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1} &= \frac{1}{1-\frac{1}{3}}\\ &= \frac{1}{\frac{2}{3}}\\ &= \frac{3}{2}\end{aligned}$$

Exercise 14

Exercise 15. Note the following alignment.

$$\begin{array}{l} 0.8\\ 0.012\\ 0.00012\\ \vdots\\ \hline 0.81212\ldots \end{array}$$

Consequently, if we ignore the first term (0.8), the remainder of the series

$$0.81212... = 0.8 + 0.012 + 0.00012 + \cdots,$$

is an infinite geometric series with *first term* $a = 0.012$ and *common ratio* $r = 0.01$. Therefore, the sum is given by

$$\begin{aligned}
S &= 0.8 + \frac{a}{1-r} \\
&= 0.8 + \frac{0.012}{1-0.01} \\
&= 0.8 + \frac{0.012}{0.99} \\
&= \frac{8}{10} + \frac{12}{990} \\
&= \frac{4}{5} + \frac{2}{165} \\
&= \frac{132}{165} + \frac{2}{165} \\
&= \frac{134}{165}
\end{aligned}$$

Exercise 15

Exercise 16. The formula for the number of permutations of n objects taken k at a time is

$$_nP_k = \frac{n!}{(n-k)!} \tag{7}$$

Then

$$\begin{aligned}
_nP_1 &= \frac{n!}{(n-1)!} \\
&= \frac{n(n-1)(n-2)\cdots 1}{(n-1)(n-2)\cdots 1} \\
&= n.
\end{aligned}$$

This can be written more concisely as follows.

$$\begin{aligned}
_nP_1 &= \frac{n!}{(n-1)!} \\
&= \frac{n(n-1)!}{(n-1)!} \\
&= n.
\end{aligned}$$

Exercise 16

Exercise 17. The formula for the number of combinations of n objects taken k at a time is

$$_nC_k = \frac{n!}{k!(n-k)!} \tag{8}$$

Then

$$\begin{aligned}
_nC_{n-2} &= \frac{n!}{(n-2)!\,(n-(n-2))!} \\
&= \frac{n!}{(n-2)!2!} \\
&= \frac{n(n-1)(n-2)(n-3)\cdots 1}{2(n-2)(n-3)\cdots 1} \\
&= \frac{n(n-1)}{2}.
\end{aligned}$$

This can be written more compactly as follows.

$$\begin{aligned} {}_nC_{n-2} &= \frac{n!}{(n-2)!\,(n-(n-2))!} \\ &= \frac{n(n-1)(n-2)!}{(n-2)!2!} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

Exercise 17

Exercise 18. There are 10 possible choices, $\{0,1,2,3,4,5,6,7,8,9\}$, for each of the digits in the social security number. Using the *Fundamental Counting Principle* the total number of social security numbers is

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^9.$$

Exercise 18

Exercise 19. Since the teachers are *not* being ranked, the order in which the instructors are chosen is not important. Therefore, we need the number of *combinations* of 60 teachers taken 3 at a time. Using Equation 8 gives

$$\begin{aligned} {}_{60}C_3 &= \frac{60!}{3!(60-3)!} \\ &= \frac{60 \cdot 59 \cdot 58 \cdot 57!}{3 \cdot 2 \cdot 1 \cdot 57!} \\ &= \frac{60 \cdot 59 \cdot 58}{3 \cdot 2} \\ &= 34,220 \end{aligned}$$

Exercise 19

Exercise 20. Now the teachers are being selected for specific positions, {chair, secretary,treasurer}, so order is important. Therefore we need the number of *permutations* of 20 teachers taken 3 at a time. Using Equation 7 gives

$$\begin{aligned} {}_{20}P_3 &= \frac{20!}{(20-3)!} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot 17!}{17!} \\ &= 6,840 \end{aligned}$$

Exercise 20

Exercise 21. The *Binomial Theorem using Factorial Notation* is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad (9)$$

Let $a = 2x$ and $b = 3y$ and expand the sum.

$$\begin{aligned}(2x+3y)^5 &= \binom{5}{0}(2x)^5(3y)^0 + \binom{5}{1}(2x)^4(3y)^1 + \binom{5}{2}(2x)^3(3y)^2 \\ &\quad + \binom{5}{3}(2x)^2(3y)^3 + \binom{5}{4}(2x)^1(3y)^4 + \binom{5}{5}(2x)^0(3y)^5 \\ &= 1(2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 \\ &\quad + 5(2x)(3y)^4 + (3y)^5 \\ &= 32x^5 + 5(16)(3)x^4y + 10(8)(9)x^3y^2 + 10(4)(27)x^2y^3 \\ &\quad + 5(2)(81)xy^4 + 243y^5 \\ &= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5\end{aligned}$$

Exercise 21

Exercise 22. The *cute* solution: The Binomial Theorem, Equation 9, can be used with $n = 5$, $a = 2$, and $b = 3$. Now

$$\begin{aligned}\sum_{k=0}^{5}\binom{5}{k}2^{5-k}3^k &= (2+3)^5 \\ &= 5^5 \\ &= 3125\end{aligned}$$

Exercise 22

Exercise 23. Again use the Binomial Theorem, Equation 9, to write

$$(a+2b)^{50} = \sum_{k=0}^{50}\binom{50}{k}a^{50-k}(2b)^k.$$

Now, the third term in the expansion is found by setting $k = 2$. *Note: The first term is found by setting $k = 0$, the second by setting $k = 1$, etc.*Consequently,

$$\begin{aligned}\text{third term} &= \binom{50}{2}a^{50-2}(2b)^2 \\ &= \frac{50!}{2!48!}a^{48}4b^2 \\ &= \frac{50\cdot 49\cdot 4}{2}a^{48}b^2 \\ &= 4900a^{48}b^2\end{aligned}$$

Exercise 23

Exercise 24. First, note that Equation 1 is true for $n = 1$.

$$\begin{aligned}1^2 &= \frac{1(1+1)(2(1)+1)}{6} \\ 1 &= \frac{1(2)(3)}{6}\end{aligned}$$

Next, assume that Equation 1 is true for $n = k$.

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \tag{10}$$

We must now show that Equation 1 is true for $n = k + 1$.

$$
\begin{aligned}
&1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&\quad = [1^2 + 2^2 + \cdots + k^2] + (k+1)^2 && \text{regroup} \\
&\quad = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{sub Equation 10} \\
&\quad = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \text{common denominator} \\
&\quad = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{simplify} \\
&\quad = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{factor out } k+1 \\
&\quad = \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{simplify} \\
&\quad = \frac{(k+1)(k+2)(2k+3)}{6} && \text{factor} \\
&\quad = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} && \text{rearrange}
\end{aligned}
$$

Therefore,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for $n = 1, 2, 3, \ldots$. Exercise 24