In Pursuit

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1 Introducing Pursuit Curves

Whether dogs chasing cats or children playing tag, pursuits are common. An interesting study can be made of the paths taken by the pursuer and the pursued during a chase. For instance, in the case of a fox chasing a rabbit, if we know the path the rabbit will take, can we predict the path the fox will take while chasing the rabbit? If we make certain assumptions about both the fox and the rabbit, using vectors and differential equations we can find that it is possible to understand the mathematics of pursuit. We can also use the mathematics to plot pursuit curves such as the ones in Figure 1.

2 A Little History

Perhaps a little history would be nice here? Where could some history be found?

3 Finding a Pursuit Curve

3.1 What We Must Assume

For simplification, we will assume that the rabbit will run on the path defined by the equation we choose. We will also assume that the fox will always run

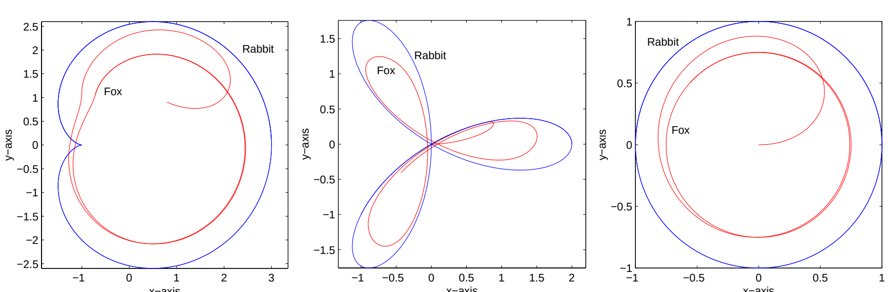

Figure 1: Pursuit Curves of a Fox and Rabbit

directly toward the rabbit. The fox runs at a speed proportional to the rabbit's speed. If the rabbit speeds up, the fox also speeds up. This has the added effect that when the rabbit stops, the fox stops also.

A different approach would be to have the fox run at a speed independent of the rabbit's speed. The equations would not be terribly different, but for our purposes we will use the former option for the development of the pursuit equations.

3.2 Equations to Pursue

We will define the position of the fox and the rabbit at time t as $\mathcal{F}(t)$ and $\mathcal{R}(t)$, respectively. Further, $\mathcal{F}(t)$ and $\mathcal{R}(t)$ will be broken into vector equations:

$$\mathcal{F}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1}$$

$$\mathcal{R}(t) = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \tag{2}$$

As shown in Equation 1, the fox's position, $\mathcal{F}(t)$, is defined by $x(t)$ and $y(t)$, which are unknown but will be solved for. Using $p(t)$ and $q(t)$—which we can define however we wish—and our assumptions about the way the fox will pursue the rabbit, we will be able to find $\mathcal{F}(t)$.

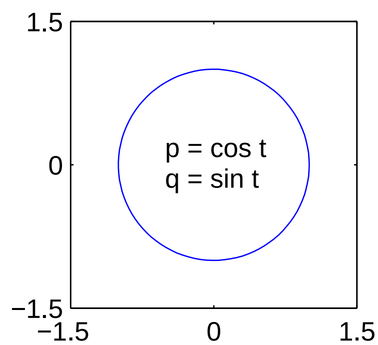

Figure 2: Unit Circle

For instance, if we say the rabbit is running on the unit circle (Figure 3.2), $p(t)$ would be $\cos t$, and $q(t)$ would be $\sin t$:

$$\mathcal{R}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{3}$$

3.3 Diving into the Mathematics

Our goal is to find $\mathcal{F}(t)$ in terms of $x(t)$ and $y(t)$. Since the fox is always heading directly toward the rabbit, the fox's velocity ($d\mathcal{F}/dt$) is in the same direction as the difference vector between the fox and the rabbit ($\mathcal{R} - \mathcal{F}$). If we use unit vectors forms of $d\mathcal{F}/dt$ and $\mathcal{R} - \mathcal{F}$, we find the pursuit curve.

First we define a new vector, $\mathcal{T}$, as a unit vector tangent to $\mathcal{V}_{\mathcal{F}}$,

$$\mathcal{T} = \frac{\mathcal{F}'}{\|\mathcal{F}'\|} \tag{4}$$

However, since the fox's speed is proportional to the rabbit's speed, Equation 4 becomes

$$\mathcal{T} = \frac{\mathcal{F}'}{k\|\mathcal{R}'\|}, \tag{5}$$

where k is a constant that we will refer to as the relative speed of the fox.

Next, we will define the second unit vector, $\mathcal{D}$, as the difference between $\mathcal{R}$ and $\mathcal{F}$ over the magnitude of $\mathcal{R} - \mathcal{F}$.

$$\mathcal{D} = \frac{\mathcal{R} - \mathcal{F}}{\|\mathcal{R} - \mathcal{F}\|} \tag{6}$$

Once again, $\mathcal{T}$ and $\mathcal{D}$ are equal because the both are unit vectors and both have the same direction. Therefore, combining Equations 5 and 6, we get:

$$\frac{\mathcal{F}'}{k\|\mathcal{R}'\|} = \frac{\mathcal{R} - \mathcal{F}}{\|\mathcal{R} - \mathcal{F}\|} \tag{7}$$

We can now solve Equation 7 for $d\mathcal{F}/dt$:

$$\mathcal{F}' = k\|\mathcal{R}'\|\frac{\mathcal{R} - \mathcal{F}}{\|\mathcal{R} - \mathcal{F}\|} \tag{8}$$

Using Equation 1 and 2, Equation 8 can be expanded:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = k\left\|\begin{pmatrix} p(t) \\ q(t) \end{pmatrix}'\right\| \frac{\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}{\left\|\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right\|} \tag{9}$$

Equation 9 can be simplified:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = k\left\|\begin{pmatrix} p'(t) \\ q'(t) \end{pmatrix}\right\| \frac{\begin{pmatrix} p(t) - x(t) \\ q(t) - y(t) \end{pmatrix}}{\left\|\begin{pmatrix} p(t) - x(t) \\ q(t) - y(t) \end{pmatrix}\right\|} \tag{10}$$

Equation 10 can be further changed:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = k\frac{\sqrt{[p'(t)]^2 + [q'(t)]^2}}{\sqrt{[p(t) - x(t)]^2 + [q(t) - y(t)]^2}} \begin{pmatrix} p(t) - x(t) \\ q(t) - y(t) \end{pmatrix} \tag{11}$$

4 Explaining the Horrific Equation

We now have a system of differential equations:

$$\begin{aligned} x'(t) &= k(p-x)\frac{\sqrt{[p'(t)]^2+[q'(t)]^2}}{\sqrt{[p(t)-x(t)]^2+[q(t)-y(t)]^2}} \\ y'(t) &= k(q-y)\frac{\sqrt{[p'(t)]^2+[q'(t)]^2}}{\sqrt{[p(t)-x(t)]^2+[q(t)-y(t)]^2}} \end{aligned} \tag{12}$$

Recall that p and q describe the path the rabbit runs on, and k is the speed of the fox relative to the rabbit. However, in most situations this system is difficult, if not impossible, to solve. For that reason, pursuit curves are best studied using numerical methods. Using a solver with the system in Equation 12, we can get a very good approximation of the pursuit curve.