Cylindrical Coordinates and MATLAB

Math 50C — Multivariable Calculus

David Arnold

David-Arnold@Eureka.redwoods.cc.ca.us

Abstract

Cylindrical coordinates are used to graph quadric surfaces. *Prequisites: Familiarity with MATLAB's* `mesh` *command and MATLAB's array operations.*

next page

close

exit

Table of Contents

title page

contents

previous page

next page

back

print doc

close

exit

Introduction

This is an interactive document designed for online viewing. We've constructed this onscreen documents because we want to make a conscientious effort to cut down on the amount of paper wasted at the College. Consequently, printing of the onscreen document has been purposefully disabled. However, if you are extremely uncomfortable viewing documents onscreen, we have provided a print version. If you click on the Print Doc button, you will be transferred to the print version of the document, which you can print from your browser or the Acrobat Reader. We respectfully request that you only use this feature when you are at home. Help us to cut down on paper use at the College.

Much effort has been put into the design of the onscreen version so that you can comfortably navigate through the document. Most of the navigation tools are evident, but one particular feature warrants a bit of explanation. The section and subsection headings in the onscreen and print documents are interactive. If you click on any section or subsection header in the onscreen document, you will be transferred to an identical location in the print version of the document. If you are in the print version, you can make a return journey to the onscreen document by clicking on any section or subsection header in the print document.

Finally, the table of contents is also interactive. Clicking on an entry in the table of contents takes you directly to that section or subsection in the document.

Working with Matlab

This document is a working document. It is expected that you are sitting in front of a computer terminal where the Matlab software is installed. You are not supposed to read this document as if it were a short story. Rather, each time your are presented with a Matlab command, it is expected that you will enter the command, then hit the Enter key to execute the command and view the result. Furthermore, it is expected that you will ponder the result. Make sure that you completely understand why you got the result you did before you continue with the reading.

Quadric Surfaces in Cylindrical Coordinates

If a point is described in cylindrical coordinates, as in **Figure 1**, the equations of transformation between cylindrical coordinates and Cartesian coordinates are as follows:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned} \tag{1}
$$

Note further that the relation between x, y, and r is given by

$$
r = \sqrt{x^2 + y^2} \tag{2}
$$

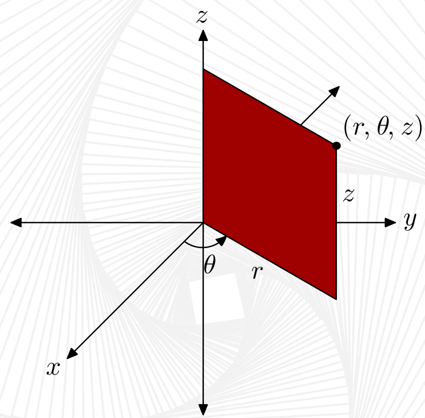

Figure 1 The geometry of cylindrical coordinates.

Matlab and Cylindrical Coordinates

You can often improve the quality of a surface drawn in MATLAB by changing to cylindrical coordinates.

Example 1

Sketch the graph of the equation $z = x^2 + y^2$.

Solution. Let's sketch the graph of $z = x^2 + y^2$ over the rectangular region $D = \{(x, y) : -1 \le x \le 1,\ -1 \le y \le 1\}$ with the following sequence of MATLAB commands.[1]

title page
contents
previous page
next page
back
print doc
close
exit

```
x=-1:.1:1;
y=-1:.1:1;
[xx,yy]=meshgrid(x,y);
zz=xx.^2+yy.^2;
mesh(xx,yy,zz);
grid on;
xlabel('x-axis')
ylabel('y-axis')
zlabel('z-axis')
title('The graph of z = x^2 + y^2')
```

These commands should produce a surface similar to that in **Figure 2**.

Although the image in **Figure 2** is nice, this particular surface is not well suited to be drawn over a rectangular region. You can get a much better representation of the surface in **Figure 2** by transforming the equation $z = x^2 + y^2$ into cylindrical coordinates.

If you substitute **equations 1** into $z = x^2 + y^2$, you arrive at the following result.

$$\begin{aligned}
z &= x^2 + y^2 \\
z &= (r\cos\theta)^2 + (r\sin\theta)^2 \\
z &= r^2\cos^2\theta + r^2\sin^2\theta \\
z &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\
z &= r^2(1) \\
z &= r^2
\end{aligned}$$

Remark 1

You can achieve the same result by substituting **equation 2** into $z = x^2 + y^2$.

[1]Remember, it is more efficient to use script files. Open MATLAB's editor and enter the commands. Save the file as `parab1.m`, then execute the script by typing `parab1` at MATLAB's prompt.

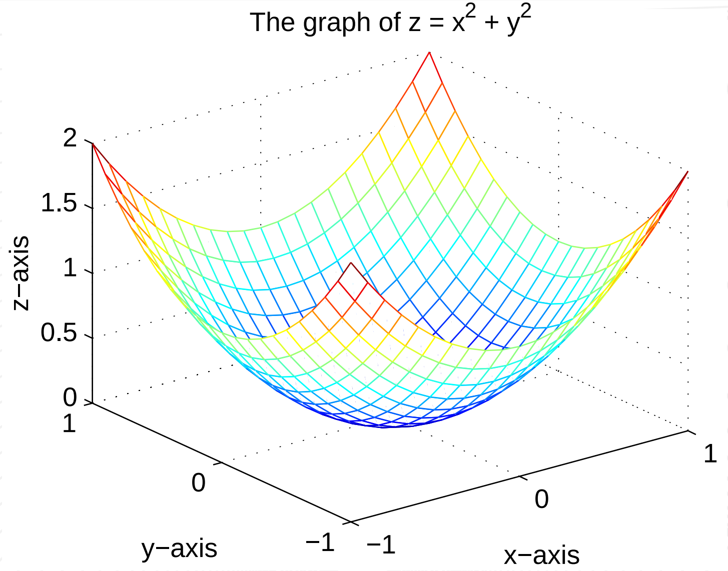

Figure 2 The graph of $z = x^2 + y^2$ over a rectangular domain.

$$z = x^2 + y^2$$

$$z = \left(\sqrt{x^2 + y^2}\right)^2$$

$$z = r^2$$

In both cases, it is important to notice that z is a function of both r and θ, even though θ is not explicitly present. In **Example 1**, z was a function of x and y. Therefore, MATLAB's `meshgrid` command was used to create a grid of (x, y) coordinate pairs. In the present case, z is a function of r and θ. Therefore, MATLAB's `meshgrid` command will be used to create a grid of (r, θ) pairs.

Once you've created a grid of (r, θ) pairs, then **equations 1** can be used to change to rectangular coordinates (x, y). Finally, z-values are computed with $z = r^2$. Note the use of array operators in all calculations.

The following MATLAB commands should produce an image similar to that in **Figure 3**.

```
R=0:.1:1;
THETA=0:pi/12:2*pi;
[r,theta]=meshgrid(R,THETA);
x=r.*cos(theta);
y=r.*sin(theta);
z=r.^2;
mesh(x,y,z);
grid on
xlabel('x-axis')
ylabel('y-axis')
zlabel('z-axis')
title('The graph of z = x^2 + y^2')
```

We will use this cylindrical coordinate approach in the remaining activities.

Complex Numbers

Complex numbers appear when you attempt to take the square root of a negative number. MATLAB's `mesh` command will fail if any input is complex. Therefore, complex numbers need special attention.

Example 2

Sketch the graph of the equation

$$z^2 - \frac{x^2}{4} - y^2 = 1 \tag{3}$$

Solution. Substitute **equations 1** into **equation 3** and solve for r.

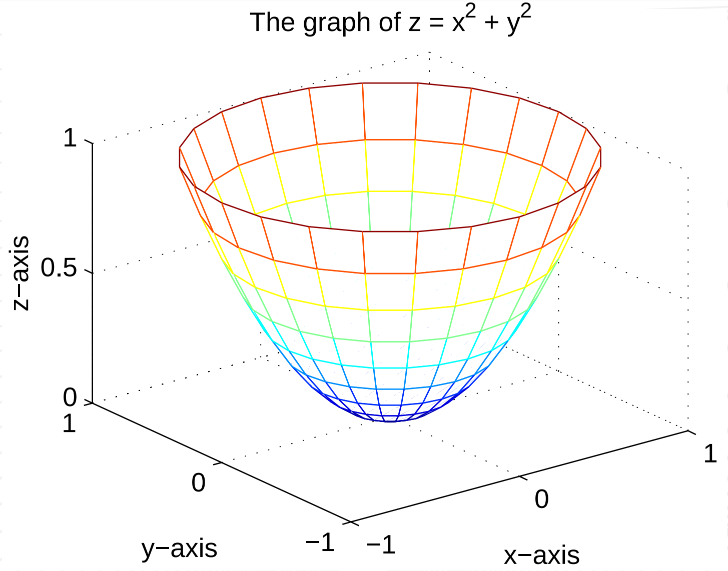

Figure 3 The graph of $z = x^2 + y^2$ in cylindrical coordinates.

$$\begin{aligned} z^2 - \frac{x^2}{4} - y^2 &= 1 \\ z^2 - \frac{(r\cos\theta)^2}{4} - (r\sin\theta)^2 &= 1 \\ 4z^2 - r^2\cos^2\theta - 4r^2\sin^2\theta &= 4 \\ 4z^2 - 4 &= r^2\left(\cos^2\theta + 4\sin^2\theta\right) \\ r &= \sqrt{\frac{4z^2 - 4}{\cos^2\theta + 4\sin^2\theta}} \end{aligned} \tag{4}$$

Note that r will be a complex number if the numerator of this last expression is negative. This can cause some difficulty in MATLAB. If any complex numbers are passed to MATLAB's `mesh` command the routine will crash and refuse to draw your surface. Therefore, you must remove all complex numbers from your matrices before passing them to the `mesh` command. You can do this by replacing each occurrence of a

title page
contents
previous page
next page
back
print doc
close
exit

complex number with a NaN (not-a-number)[2].

```
>> help NaN

NaN    Not-a-Number.
   NaN is the IEEE arithmetic representation for Not-a-Number.
   A NaN is obtained as a result of mathematically undefined
   operations like 0.0/0.0  and inf-inf.

   See also INF.
```

Because r is a function of z and θ, let's begin by creating a mesh of z and θ-values. We will work at the command line in this example.

```
>> THETA=0:pi/12:2*pi;
>> Z=-2:.1:2;
>> [z,theta]=meshgrid(Z,THETA);
```

Use **equation 4** to compute the r-values.

```
>> r=sqrt((4*z.^2-4)./(cos(theta).^2+4*sin(theta).^2));
```

MATLAB has a routine that will check if your matrix has any complex entries.

```
>> help isreal

ISREAL True for real array.
   ISREAL(X) returns 1 if all elements in X have zero
   imaginary part and 0 otherwise.
```

Check if matrix r has any complex entries.

[2] MATLAB is case-sensitive. You must enter uppercase-N, lowercase-a, and uppercase-N.

title page
contents
previous page
next page
back
print doc
close
exit

```
>> isreal(r)
ans =
     0
```

The response 0 indicates that not all of the entries of the matrix r are real numbers. You can find the complex entries with MATLAB's `find` command.

```
>> help find

FIND   Find indices of nonzero elements.
   I = FIND(X) returns the indices of the vector X that are
   non-zero. For example, I = FIND(A>100), returns the indices
   of A where A is greater than 100.
```

If $z = a + bi$, the real part of z is a and the imaginary part of z is b. In symbols, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. A number z is real if and only if $\mathrm{Re}(z) = z$; otherwise, the number is complex.

```
>> help real

REAL   Complex real part.
   REAL(X) is the real part of X.
   See I or J to enter complex numbers.
```

Now, let's find which entries of the matrix r are complex numbers.

```
>> k=find(real(r)~=r);
```

The symbol ~= is MATLAB's symbol for $\neq$ (not equals). The vector k now contains the index of each complex entry of the matrix r. We want to replace each of these entries with a NaN.[3]

```
>> r(k)=NaN;
```

[3] In MATLAB 4, enter `r(k)=NaN*ones(size(k)` instead. This last command deserves a few words of explanation. First create a vector of 1's the same size as the vector k with the command `ones(size(k))`. Multiply each entry of this vector by NaN and you have a vector of NaN's the same size as the vector k. Finally, `r(k)=NaN*ones(size(k))` will replace the entries of the matrix r with indices k with NaN's.

Now use the **transformations 1** to change from cylindrical coordinates to Cartesian coordinates.

```
>> x=r.*cos(theta);
>> y=r.*sin(theta);
```

The following command will produce an image similar to that in **Figure 4**.

```
>> mesh(x,y,z)
```

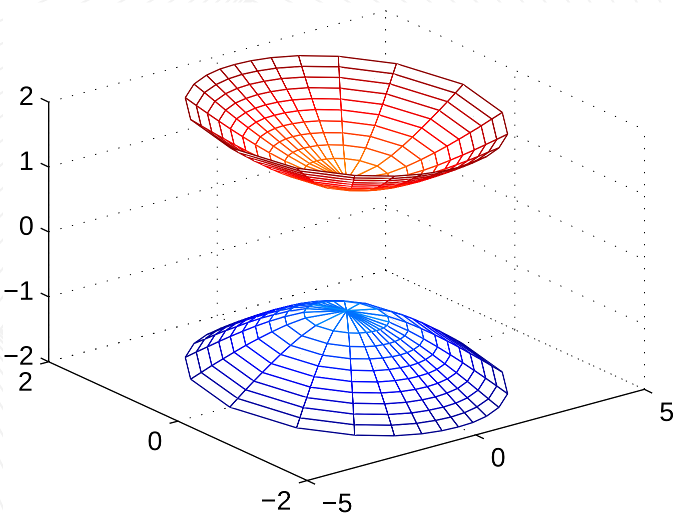

Figure 4 A hyperboloid with two sheets.

Add labels and a title and, more importantly, scale the axes equally[4] to remove distortion and see that cross sections of the surface parallel to the xy-plane are more elliptical than circular.

```
>> xlabel('x-axis')
>> ylabel('y-axis')
```

[4] You might also want to investigate the command `axis equal`.

```
>> zlabel('z-axis')
>> title('The graph of z^2 - x^2/4 - y^2 = 1')
>> grid on
>> axis([-4 4 -4 4 -2 2])
```

These commands produce an image similar to that in **Figure 5**.

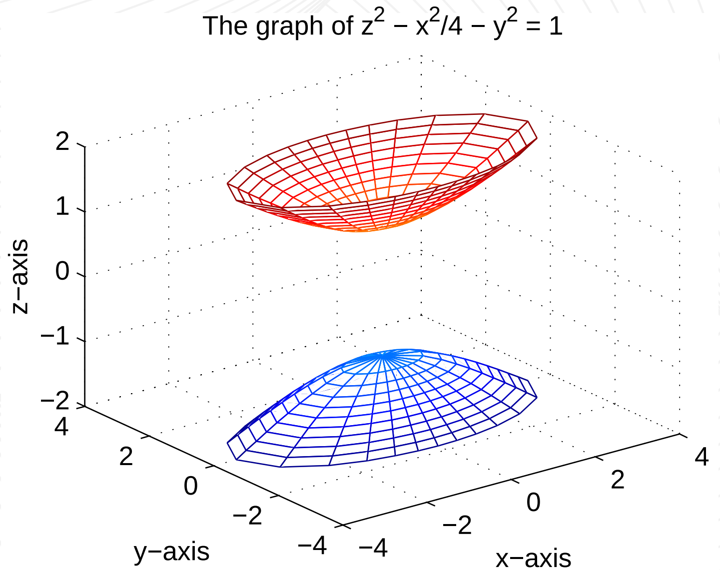

Figure 5 Formatting the image.

Advance Planning

With a little careful planning, complex numbers can be avoided. Let's try a different approach on the graph of equation of **example 2**.

Example 3

Sketch the graph of the equation

$$z^2 - \frac{x^2}{4} - y^2 = 1. \tag{5}$$

Once again, substituting **equations 1** into **equation 5** yields the following result:

$$r = \sqrt{\frac{4z^2 - 4}{\cos^2\theta + 4\sin^2\theta}} \tag{6}$$

You will introduce complex numbers only if you attempt to take the square root of a negative number in **equation 6**. With a little advance planning, complex numbers can be avoided. Since the denominator of the expression underneath the radical in **equation 6** must always be positive, r will be complex only if the numerator $4z^2 - 4$ is negative. So, what values of z make the expression $4z^2 - 4$ non-negative?

$$\begin{aligned} 4z^2 - 4 &\geq 0 \\ 4z^2 &\geq 4 \\ z^2 &\geq 1 \\ \sqrt{z^2} &\geq \sqrt{1} \\ |z| &\geq 1 \end{aligned}$$

But $|z| \geq 1$ only if $z \leq -1$ or $z \geq 1$. Therefore, if we restrict the values of z so that they are less than or equal to -1 or greater than or equal to 1, r will be a real number. The following MATLAB code was used to produce the image in **Figure 6**.

```
z=[-3:.2:-1,1:.2:3];
theta=linspace(0,2*pi,36);
[z,theta]=meshgrid(z,theta);
r=sqrt((4*z.^2-4)./(cos(theta).^2+4*sin(theta).^2));
x=r.*cos(theta);
y=r.*sin(theta);
mesh(x,y,z)
axis equal
title('z^2-x^2/4-y^2=1')
xlabel('x-axis')
ylabel('y-axis')
zlabel('z-axis')
```


title page
contents
previous page
next page
back
print doc
close
exit

The most important command in this sequence is the command `z=[-3:.2:-1,1:.2:3]`. This command crafts a vector with values from -3 to -1, incremented by 0.2, then adds numbers from 1 to 3, incremented again by 0.2. This ensures that only z-values less that or equal to -1 or greater than or equal to 1 are used, avoiding the complex number issue posed in **Example 2**. Obviously, this is a much better approach. It should be the approach you use in the exercises, should complex numbers need to be avoided.

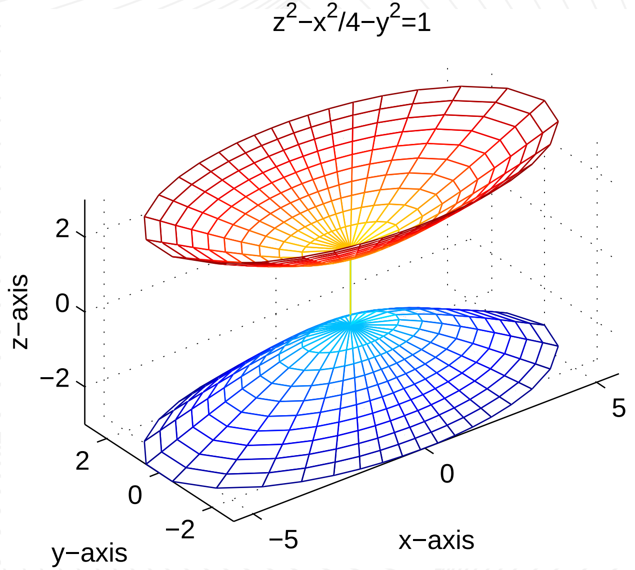

Figure 6 Advance planning avoids complex numbers.

Homework Exercises

1. Use the technique of **example 1** in this activity to draw the graph of the surfaces defined by each of the following equations. Label the axes and provide an appropriate title. Obtain a printout of your result and hand in during the next class period.

a. $z = 9 - x^2 - y^2$

b. $z = \sqrt{x^2 + y^2}$

2. Use the technique of **example 2** and **example 3** (remember, advance planning can eliminate complex numbers) to draw the graph of the surfaces defined by each of the following equations. Label the axes, provide an appropriate title, and obtain a printout of your result to hand in during the next class period.

a. $z = \sqrt{1 - x^2 - y^2}$

b. $z^2 - x^2 - y^2 = 4$

c. $x^2 + \dfrac{y^2}{4} - z^2 = 1$

d. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

title page
contents
previous page
next page
back
print doc
close
exit