Curvature
in
MATLAB

Math 50C — Multivariable Calculus

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Abstract

In this activity you will the concept of curvature. MATLAB's `Symbolic Toolbox` is used to simplify the tedious differentiation requirement. *Prequisites: Some familiarity with MATLAB's array operations and* `plot` *command is useful.*

1 Introduction

This is an interactive document designed for online viewing. We've constructed this onscreen documents because we want to make a conscientious effort to cut down on the amount of paper wasted at the College. Consequently, printing of the onscreen document has been purposefully disabled. However, if you are extremely uncomfortable viewing documents onscreen, we have provided a print version. If you click on the Print Doc button, you will be transferred to the print version of the document, which you can print from your browser or the Acrobat Reader. We respectfully request that you only use this feature when you are at home. Help us to cut down on paper use at the College.

Much effort has been put into the design of the onscreen version so that you can comfortably navigate through the document. Most of the navigation tools are evident, but one particular feature warrants a bit of explanation. The section and subsection headings in the onscreen and print documents are interactive. If you click on any section or subsection header in the onscreen document, you will be transferred to an identical location in the print version of the document. If you are in the print version, you can make a return journey to the onscreen document by clicking on any section or subsection header in the print document.

Finally, the table of contents is also interactive. Clicking on an entry in the table of contents takes you directly to that section or subsection in the document.

1.1 Working with Matlab

This document is a working document. It is expected that you are sitting in front of a computer terminal where the MATLAB software is installed. You are not supposed to read this document as if it were a short story. Rather, each time your are presented with a MATLAB command, it is expected that you will enter the command, then hit the Enter key to execute the command and view the result. Furthermore, it is expected that you will ponder the result. Make sure that you completely understand why you got the result you did before you continue with the reading.

2 The Unit Tangent Vector

Most multivariable calculus syllabi include some discussion of *curvature*, a number that indicates the amount of bending a curve does at each point along it path. Students happily memorize formulae and calculate all the derivatives required, but oftentimes no connection is made to the rich geometry that lies at the core of the topic. In this activity, we will rectify this problem.

We will concentrate on curves in two-space (the real plane), but by simply adding a third dimension, our comments easily apply to three-space. That said, let's begin with the concept of a *position vector*.

In **Figure 1**, the tip of the position vector $\mathbf{r}(t) =< x(t), y(t) >$ traces out a path in the plane with the passage of time. Here we are assuming that the parameter t represents time.

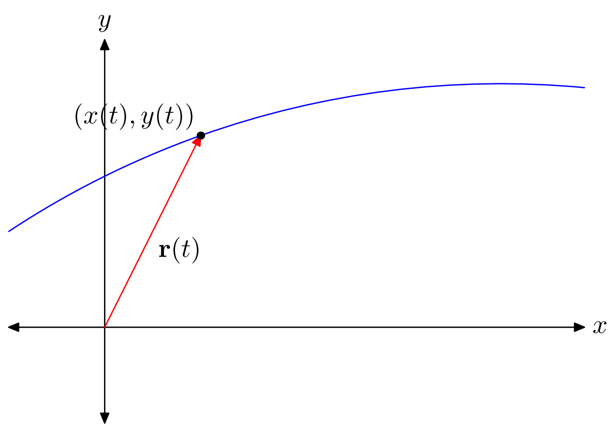

Figure 1 The position vector.

Recall that in your first calculus course that the velocity is the instantaneous rate at which the position changes with respect to time. It is no different with vector valued functions of time. That is,

$$\mathbf{v}(t) = \mathbf{r}'(t) =< x'(t), y'(t) > .$$

An important distinction is the fact that the velocity is a vector. Its magnitude is the instantaneous speed and it points in the direction of motion. Note that the velocity vector shown in **Figure 2** is tangent to the path at the tip of the position vector.

If we divide the velocity vector by its length, then we get the unit tangent vector

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

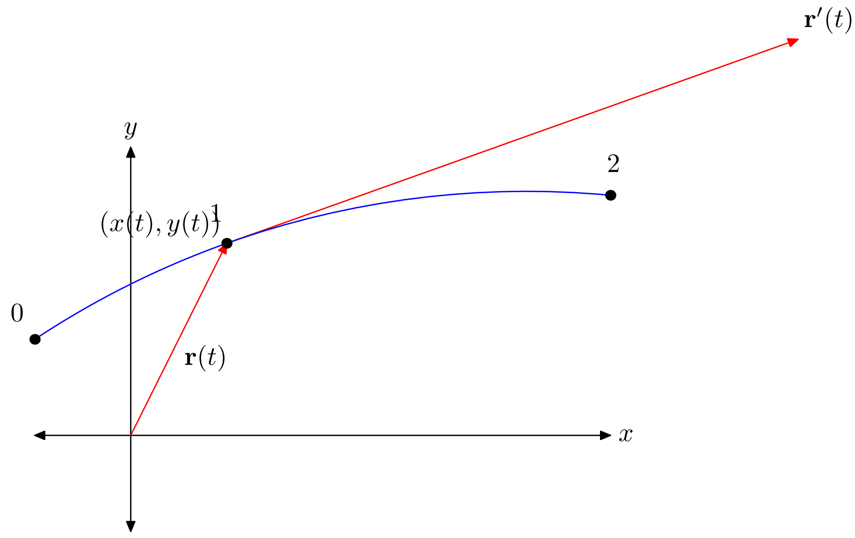

Figure 2 The position and velocity vectors.

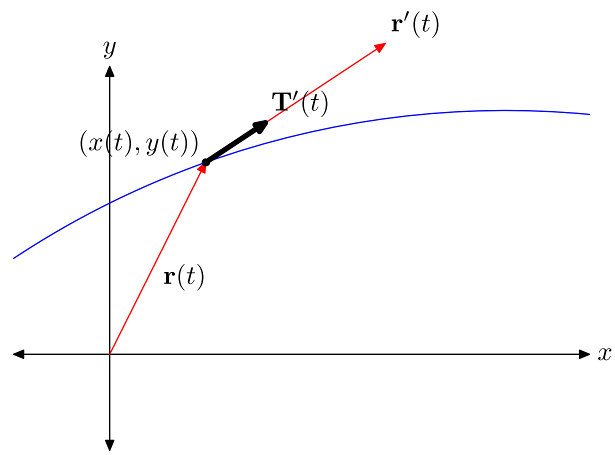

Figure 3 The unit tangent vector points in the direction of the velocity vector, $\mathbf{v}(t) = \mathbf{r}'(t)$.

a vector of length one that points in the direction of velocity. The unit tangent vector is shown in **Figure 3**.

Now, let's examine the unit tangent vector in MATLAB. Consider the path defined by the vector value function

$$\mathbf{r}(t) =< t, t^2 >$$

We plot this over the time interval $[-2, 2]$ with the following MATLAB commands. The output is shown in **Figure 4**.

```
t=linspace(-2,2);
x=t; y=t.^2;
plot(x,y)
```

Calculate the derivative of $\mathbf{r}(t)$ to find the velocity.

$$\mathbf{r}'(t) = < 1, 2t >$$

Calculate the magnitude of the velocity.

$$\|\mathbf{r}'(t)\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

Divide the velocity by its magnitude to find the unit tangent vector.

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{< 1, 2t >}{\sqrt{1 + 4t^2}} \\ &= \left\langle \frac{1}{\sqrt{1 + 4t^2}}, \frac{2t}{\sqrt{1 + 4t^2}} \right\rangle \end{aligned}$$

MATLAB's `quiver` command is used to draw vectors. The syntax is `quiver(x,y,vx,vy,s)`, where `x` and `y` are the coordinates of the tails of the vectors whose components are stored in the vectors `vx` and `vy`. The argument `s` is a scaling factor. Setting $s = 0$ turns off all scaling and plots the vectors at their actual length. The following MATLAB commands were used to plot the unit tangent vectors displayed in **Figure 4**.

```
t=-2:1:2;
x=t; y=t.^2;
xT=1./sqrt(1+4*t.^2);
yT=2*t./sqrt(1+4*t.^2);
quiver(x,y,xT,yT,0)
```

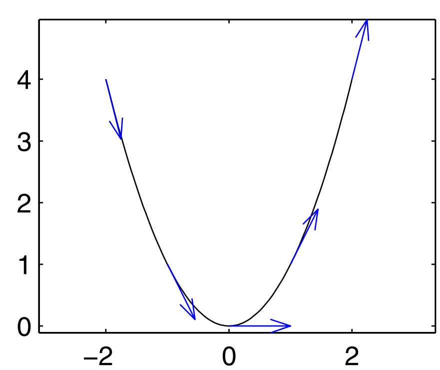

Figure 4 Unit tangent vectors along the path defined by $\mathbf{r}(t) = < t, t^2 >$.

It is important to note that each of the unit tangent vectors shown in **Figure 4** have length one and point in the direction of the velocity.

If we look at the derivative of the unit tangent $\mathbf{T}$, then naturally we are examining $d\mathbf{T}/dt$, the rate at which the unit tangent vector is changing with respect to time. Clearly, this has to have something to do with the curvature at each point on the path defined by $\mathbf{r}(t) =< t, t^2 >$ (and, more generally, on the path defined by $\mathbf{r}(t) =< x(t), y(t) >$).

If the unit tangent vector is changing rapidly with respect to time, then one would expect a great deal of bending of the curve at that point. Conversely, if the unit tangent vector is changing slowly with respect to time, then one would not expect a great deal of bending. Obviously, the derivative of the unit tangent vector has an intimate connection with the "curvature" at each point on the path.

So, which way does the the vector $\mathbf{T}'(t)$ point? And how long is it? To answer the first of these questions, remember that the length of the unit tangent vector is 1.

$$\|\mathbf{T}\| = 1$$

Of course, this means that the square of the magnitude is also one, and if one remembers that the fact that $\|\mathbf{T}\|^2 = \mathbf{T} \cdot \mathbf{T}$, then we can write

$$\begin{aligned} \|\mathbf{T}\|^2 &= 1 \\ \mathbf{T} \cdot \mathbf{T} &= 1. \end{aligned}$$

Differentiate with respect to t.

$$\begin{aligned} \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' &= 0 \\ 2\mathbf{T} \cdot \mathbf{T}' &= 0 \\ \mathbf{T} \cdot \mathbf{T}' &= 0 \end{aligned}$$

This result tells us that the unit tangent vector $\mathbf{T}$ is perpendicular to its derivative $\mathbf{T}'$. But, as you can see in **Figure 4**, this still leaves us with two choices for the direction of $\mathbf{T}'$. How do we choose?

Recall that the unit tangent for the curve defined by $\mathbf{r}(t) =< t, t^2 >$ was given by

$$T = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle.$$

A somewhat lengthy calculation reveals that the derivative of the unit tangent vector is

$$T' = \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle.$$

The `Symbolic Toolbox`, MATLAB's interface to Maple, relieves the user from the drudgery of this calculation. First, declare `t` as a real symbolic variable, define `r`, and take its derivative.[1]

[1] Although these calculations were carried out at the command line, it is much more efficient to put them into a script file.

```
>> syms t real
>> r=[t,t^2]
>> rp=diff(r)
```

Next, the length of $\mathbf{r}'(t)$ is calculated with $\|\mathbf{r}'(t)\|^2 = \mathbf{r}'(t) \cdot \mathbf{r}'(t)$.

```
>> nrp=sqrt(dot(rp,rp))
```

The unit tangent is found with $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$.

```
>> T=rp/nrp
T =
[   1/(1+4*t^2)^(1/2)]
[ 2*t/(1+4*t^2)^(1/2)]
```

Finally, differentiate this last result to find $\mathbf{T}'(t)$.

```
>> Tp=diff(T)
Tp =
[                    -4/(1+4*t^2)^(3/2)*t]
[ 2/(1+4*t^2)^(1/2)-8*t^2/(1+4*t^2)^(3/2)]
```

Computer algebra systems (CAS) have the terrible habit of outputting some ghastly responses. MATLAB's `simple` command attempts to find the most compact form of an answer. That is, it simplifies your answer.

```
>> Tp=simple(Tp)
Tp =
[ -4/(1+4*t^2)^(3/2)*t]
[    2/(1+4*t^2)^(3/2)]
```

The MATLAB command `pretty` attempts to put the output in human readable form. We take the transpose to save space.

```
>> pretty(Tp')

                    [         t~                 2       ]
                    [-4 --------------    --------------]
                    [             2 3/2             2 3/2]
                    [   (1 + 4 t~ )       (1 + 4 t~ )    ]
```

This agrees nicely with our hand calculated result. The tilde in `t~` is Maple's way of tagging a variable that has assumptions associated with that variable. When we issued the command `syms t real`, we issued an assumption that `t` was a real variable. The tilde in the output reminds us that we have made this assumption.

We now plot the derivatives of the unit tangent vectors in **Figure 4**. The result of the following set of commands is shown in **Figure 5**.

```
t=-2:1:2;
x=t;y=t.^2;
xTp=-4*t./(1+4*t.^2).^(3/2);
yTp=2./(1+4*t.^2).^(3/2);
quiver(x,y,xTp,yTp,0,'r')
```

Note that each instance of $\mathbf{T}'$ is orthogonal to the unit tangent vector. Further, note that each instance of $\mathbf{T}'$ points inward, in the direction in which the unit tangent vector is turning.

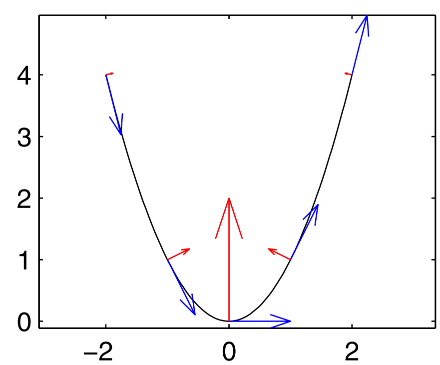

Figure 5 The derivative of the unit tangent points inward.

Note that $\mathbf{T}'(0)$, attached at the point $(0,0)$, is much longer than other instances of $\mathbf{T}'$. This is, of course, because the unit tangent vector is changing more rapidly at $t = 0$, and $\mathbf{T}'(0)$ is the rate at which the unit tangent vector is changing with respect to time at $t = 0$. This clearly explains why the "curvature" is most extreme at $t = 0$. Conversely, where the unit tangent is hardly changing at all, the magnitude of $\mathbf{T}'$ is quite small; that is, the vector $\mathbf{T}'$ is rather short. This is easily seen at the extreme ends of the parabola in **Figure 5**.

However, to pursue a proper definition of curvature, we need to reparametrize our curves in terms of arc length s.

3 Arc Length

Suppose that the position of a particle in space is given as a vector valued function of time.

$$\mathbf{r}(t) =< \cos t, \sin t, t > \tag{1}$$

The path of the particle is easily sketched with the assistance of MATLAB.

```
t=linspace(0,4*pi);
x=cos(t);y=sin(t);z=t;
plot3(x,y,z)
grid on
```

This set of commands draws two cycles of the helix shown in **Figure 1**.

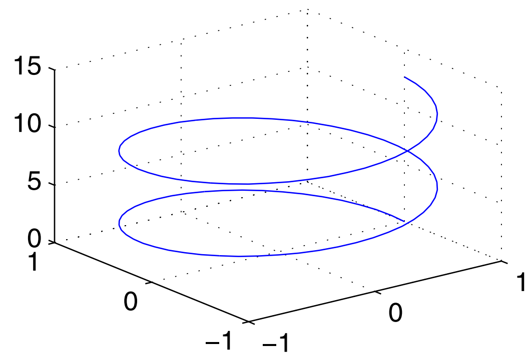

Figure 6 A cylindrical helix.

What we will do next is reparametrize the helical path in terms of arc length. We denote the arc length by the variable s and we seek to find the length of the helical path travelled between the times $\tau = 0$ and $\tau = t$.

During a small increment of time $d\tau$, the speed $\|\mathbf{v}(\tau)\|$ remains relatively constant. The distance travelled during this time is simply the product of the speed and the amount of time that has passed. In symbols, $ds = \|\mathbf{v}(\tau)\|\, d\tau$. To find the total distance travelled, we integrate over the interval $[0, t]$.

The position of the particle is given by

$$\mathbf{r}(t) = < \cos t, \sin t, t > .$$

Therefore, the first derivative determines the velocity.

$$\mathbf{v}(t) = \mathbf{r}'(t) = < -\sin t, \cos t, 1 >$$

The speed is the magnitude of the velocity.

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

Consequently, the arc length s, as a function of t, is determined by the following calculation.

$$\begin{aligned} s &= \int_0^t \|\mathbf{v}(\tau)\|\, d\tau \\ &= \int_0^t \sqrt{2}\, d\tau \\ &= \sqrt{2}t \end{aligned}$$

Solve this last equation for t to get $t = s/\sqrt{2}$. Making this substitution, **equation 1** becomes

$$\mathbf{r}(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle. \tag{2}$$

It is important to understand that the path has not changed, only the parametrization of that path. This is easily demonstrated with the following MATLAB code. Note that as t varies from 0 to 4π, s varies from 0 to $4\pi\sqrt{2}$. The following commands produce a path **Figure 2** that is the duplicate of the path shown in **Figure 1**.

```
figure
s=linspace(0,4*pi*sqrt(2));
x=cos(s/sqrt(2));y=sin(s/sqrt(2));z=s/sqrt(2);
plot3(x,y,z,'k')
ax=axis;
grid on
```

3.1 Why Bother?

This seems to be a lot of computational fuss just to draw the same helical path. However, the true value of this reparametrization is not discovered until you take the derivative of the position vector with respect to s.

$$\mathbf{v}(s) = \mathbf{r}'(s) = \left\langle -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Again, this is the velocity of the particle, but this time in terms of s. Adding velocity vectors to the path for various values of s reveals that the velocity always has unit length, giving the reparametrization the name "unit speed curve."

```
hold on
s=4*pi*sqrt(2)*(0:7)/8;
x=cos(s/sqrt(2));y=sin(s/sqrt(2));z=s/sqrt(2);
xT=-1/sqrt(2)*sin(s/sqrt(2));
yT=1/sqrt(2)*cos(s/sqrt(2));
zT=1/sqrt(2)*ones(size(s));
quiver3(x,y,z,xT,yT,zT,0,'b');
axis(ax)
```

This code provides remarkable evidence in **Figure 2** that the curve has unit speed at each position on its path. Of course, it is an easy calculation to demonstrate that each velocity vector has unit length.

$$
\begin{aligned}
\|\mathbf{v}(s)\| &= \sqrt{\frac{1}{2}\sin^2\frac{s}{\sqrt{2}} + \frac{1}{2}\cos^2\frac{s}{\sqrt{2}} + \frac{1}{2}} \\
&= \sqrt{\frac{1}{2}\left(\sin^2\frac{s}{\sqrt{2}} + \cos^2\frac{s}{\sqrt{2}}\right) + \frac{1}{2}} \\
&= \sqrt{\frac{1}{2} + \frac{1}{2}} \\
&= 1
\end{aligned}
$$

This is a distinct advantage. If you parametrize a curve in terms of arc length, the derivative of the position is a *unit tangent vector*. That is,

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}.$$

As before, if we differentiate the unit tangent vector with respect to s, then we will know that rate at which the unit tangent vector is changing with respect to s. This gives us some notion of the "curvature" at each position on the path. In the case of the helix,

$$\mathbf{T}(s) = \left\langle -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle,$$

so

$$\mathbf{T}'(s) = \left\langle -\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right\rangle.$$

These are added to the image in **Figure 2** with the following code.

```
xTp=-1/2*cos(s/sqrt(2));
yTp=-1/2*sin(s/sqrt(2));
zTp=zeros(size(yTp));
quiver3(x,y,z,xTp,yTp,zTp,0,'r')
```

Again, $\|\mathbf{T}\| = 1$, which leads to $\mathbf{T}\cdot\mathbf{T} = 1$ and $\mathbf{T}\cdot\mathbf{T}' = 0$. Thus, $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal at each position on the path. In **Figure 2**, each instance of $\mathbf{T}'$ seems to have the same length. This seems reasonable, as the path is bending in an almost circular manner, making the "curvature" equal at all positions on the path.

All of this preparation leads us to the standard definition of curvature.

4 Curvature

At each point of a unit speed curve, we assign a number called the *curvature*, which is an indicator of the amount of bending of the curve that occurs at that position. For example, if the curvature at every point of the path is zero, then the path is clearly a line, as no "bending" is occurring

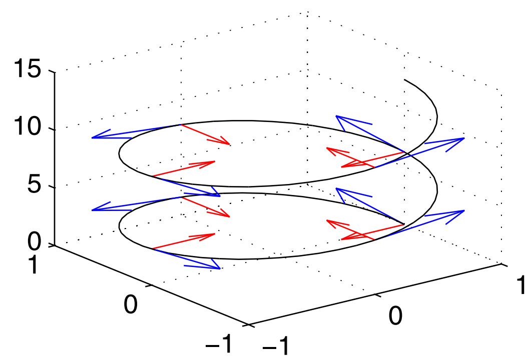

Figure 7 Each velocity vector on the "unit speed curve" has unit length.

whatsoever. If the curvature at each point of path is constant, then we imagine some kind of circular motion as the bending is uniform at each point on the path. If the curvature is a large number, then we imagine a drastic, maybe even "dangerous bend" in the path.

Clearly, the curvature is related to the rate at which the unit tangent vector is changing with respect to s. If the unit tangent vector is turning drastically, then we've got a lot of bending going on and the curvature is a very large number. If the unit tangent vector is turning only a little bit, then the bending is minimal and we have a low number for the curvature at that position on the path.

Therefore, the following definition of curvature seems logical; it is the magnitude of derivative of the unit tangent vector with respect to s. In symbols,

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$