(1a) 5pts

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \int_0^1 \frac{dx}{\sqrt{x}(x+4)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+4)}$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}(x+4)} + \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}(x+4)}$$

I will first find an antiderivative:

$$\int \frac{dx}{\sqrt{x}(x+4)}$$

$x = u^2$
$dx = 2u\,du$

$$= \int \frac{2u\,du}{\sqrt{u^2}(u^2+4)}$$

$u = 2\tan\theta$
$du = 2\sec^2\theta\,d\theta$

$$= \int \frac{2\,du}{u^2+4}$$

$$= \int \frac{2(2\sec^2\theta\,d\theta)}{4\tan^2\theta+4}$$

$$= \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+1}$$

$$= \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$$

$$= \int d\theta$$

$$= \theta$$

$$= \tan^{-1} \frac{u}{2}$$

$$= \tan^{-1} \frac{\sqrt{x}}{2}$$

$$u = 2\tan\theta$$

$$\tan\theta = \frac{u}{2}$$

$$\theta = \tan^{-1} \frac{u}{2}$$

$$x = u^2 \Rightarrow u = \sqrt{x}$$

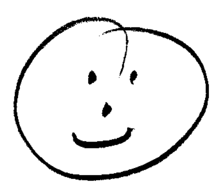
Check by differentiation:

$$D_x \tan^{-1} \frac{\sqrt{x}}{2} = \frac{1}{1 + \left(\frac{\sqrt{x}}{2}\right)^2} D_x \frac{\sqrt{x}}{2}$$

$$= \frac{1}{1 + \frac{x}{4}} \frac{1}{4} x^{-1/2}$$

$$= \frac{1}{1 + \frac{x}{4}} \cdot \frac{1}{4\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}(4 + x)}$$



Now, $\displaystyle\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}(x+4)} = \lim_{a\to 0^+} \tan^{-1}\frac{\sqrt{x}}{2}\Big|_a^1$

$$= \lim_{a\to 0^+}\left[\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{\sqrt{a}}{2}\right]$$

$$= \tan^{-1}\frac{1}{2} - 0$$

$$= \tan^{-1}\frac{1}{2}$$

Secondly, $\displaystyle\lim_{b\to \infty} \int_1^b \frac{dx}{\sqrt{x}(x+4)} = \lim_{b\to \infty} \tan^{-1}\frac{\sqrt{x}}{2}\Big|_1^b$

$$= \lim_{b\to\infty}\left[\tan^{-1}\frac{\sqrt{b}}{2} - \tan^{-1}\frac{1}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{1}{2}$$

Hence,

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \tan^{-1}\frac{1}{2} + \left(\frac{\pi}{2} - \tan^{-1}\frac{1}{2}\right)$$

$$= \pi/2$$

5pts

1b

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{a \to 2^-} \int_1^a \frac{dx}{(x-2)^{2/3}} + \lim_{a \to 2^+} \int_a^4 \frac{dx}{(x-2)^{2/3}}$$

$$= I_1 + I_2$$

We need an antiderivative.

$$\int \frac{dx}{(x-2)^{2/3}} = \int (x-2)^{-2/3}\, dx$$

$$= 3(x-2)^{1/3}$$

Check: $D_x\, 3(x-2)^{1/3} = 3 \cdot \frac{1}{3} \cdot (x-2)^{-2/3}$

$$= (x-2)^{-2/3}$$

Hence,

$$I_1 = \lim_{a \to 2^-} \int_1^a \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{a \to 2^-} 3(x-2)^{1/3} \Big|_1^a$$

$$= \lim_{a \to 2^-} \left[3(a-2)^{1/3} - 3(1-2)^{1/3} \right]$$

$$= 0 + 3$$

$$= 3$$

Secondly,

$$I_2 = \lim_{a \to 2^+} \int_a^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{a \to 2^+} 3(x-2)^{1/3} \Big|_a^4$$

$$= \lim_{a \to 2^+} \left[3(4-2)^{1/3} - 3(a-2)^{1/3} \right]$$

$$= 3 \cdot 2^{1/3} - 0$$

$$= 3 \cdot 2^{1/3} \text{ or } 3\sqrt[3]{2}$$

Thus,

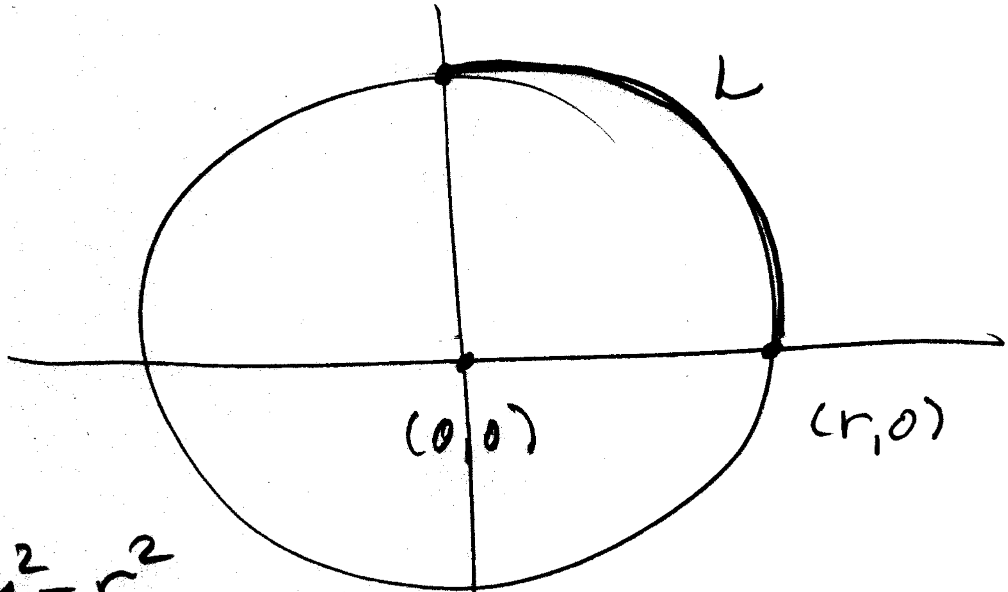
$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = I_1 + I_2$$

$$= 3 + 3\sqrt[3]{2}$$

$$= 3(1 + \sqrt[3]{2})$$

(2)

5 pts



$x^2+y^2=r^2$

$y^2=r^2-x^2$

$y=\sqrt{r^2-x^2}$ ← choose + because top half of circle

$dy' = \frac{1}{2}(r^2-x^2)^{-1/2}(-2x)$

$y' = -x(r^2-x^2)^{-1/2}$

$(y')^2 = x^2(r^2-x^2)^{-1} = \dfrac{x^2}{r^2-x^2}$

$$ds = \sqrt{1+(y')^2}\, dx$$

$$= \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$$

$$= \sqrt{\frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}}\, dx$$

$$= \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$

$$= \frac{r}{\sqrt{r^2-x^2}}\, dx$$

$$L = \int_0^r ds = \int_0^r \frac{r}{\sqrt{r^2-x^2}}\, dx$$

Note that as $x \to r$, the denominator $\to 0$.
Hence, this integral is improper. Hence,

$$L = \lim_{t \to r^-} \int_0^t \frac{r}{\sqrt{r^2-x^2}}\, dx.$$

We need an antiderivative. Trig sub holds the solution.

$$\int \frac{r\,dx}{\sqrt{r^2-x^2}} \qquad x = r\sin\theta,\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$
$$dx = r\cos\theta\,d\theta$$

$$= \int \frac{r(r\cos\theta\,d\theta)}{\sqrt{r^2-r^2\sin^2\theta}}$$

$$= \int \frac{r^2\cos\theta\,d\theta}{r\sqrt{1-\sin^2\theta}}$$

$$= r\int \frac{\cos\theta\,d\theta}{\sqrt{\cos^2\theta}}$$

$$= r\int \frac{\cos\theta\,d\theta}{|\cos\theta|}, \text{ because } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$= r\int \frac{\cos\theta\,d\theta}{\cos\theta} \qquad \boxed{\begin{aligned}\sin\theta &= \frac{x}{r}\\ \theta &= \sin^{-1}\frac{x}{r}\end{aligned}}$$

$$= r\int d\theta = r\theta = r\sin^{-1}\frac{x}{r}$$

Hence,

$$L = \lim_{t \to r^-} r \sin^{-1}\frac{x}{r}\Big|_0^t$$

$$= \lim_{t \to r^-} \left[r\sin^{-1}\frac{t}{r} - r\sin^{-1}\frac{0}{r} \right]$$

$$= \lim_{t \to r^-} \left[r\sin^{-1}\frac{t}{r} \right]$$

$$= r\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi r}{2}$$

Hence, the total circumference is

$$C = 4L = 4\left(\frac{\pi r}{2}\right) = 2\pi r,$$

which is the "classic" well known formula for the circumference of a circle.

5pts

③ $\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st}\sin t\,dt$

We need an antiderivative.

$\int e^{-st}\sin t\,dt$

	D	I
+	e^{-st}	$\sin t$
−	$-se^{-st}$	$-\cos t$
+	s^2e^{-st}	$-\sin t$

Hence,

$$\int e^{-st}\sin t\,dt = -e^{-st}\cos t + se^{-st}\sin t - s^2\int e^{-st}\sin t\,dt$$

$$\int e^{-st}\sin t\,dt + s^2\int e^{-st}\sin t\,dt = -e^{-st}[\cos t + s\sin t]$$

$$(1+s^2)\int e^{-st}\sin t\,dt = -e^{-st}[\cos t + s\sin t]$$

$$\int e^{-st}\sin t\,dt = \frac{-e^{-st}}{1+s^2}[\cos t + s\sin t]$$

Now,

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt$$

$$= \lim_{T\to\infty} \int_0^T e^{-st} \sin t \, dt$$

$$= \lim_{T\to\infty} \left[\frac{-e^{-st}}{1+s^2} (\cos t + s \sin t) \right]_0^T$$

$$= \lim_{T\to\infty} \left[\frac{-e^{-sT}}{1+s^2} (\cos T + s \sin T) + \frac{e^{-s(0)}}{1+s^2} [\cos(0) + s \sin(0)] \right]$$

$$= \lim_{T\to\infty} \left[\frac{-e^{-sT}}{1+s^2} (\cos T + s \sin T) + \frac{1}{1+s^2} \right]$$

$$= \frac{1}{1+s^2}, \text{ provided } s > 0.$$

Note: ~~In~~ the limit

$$\lim_{T \to \infty} \left[\frac{-e^{-sT}}{1+s^2} (\cos T + s \sin T) \right],$$

as T gets large, the cosine and sine terms oscillate, but importantly, they are bounded by their amplitudes. If $s > 0$, then $e^{-sT} \to 0$ as $T \to \infty$.

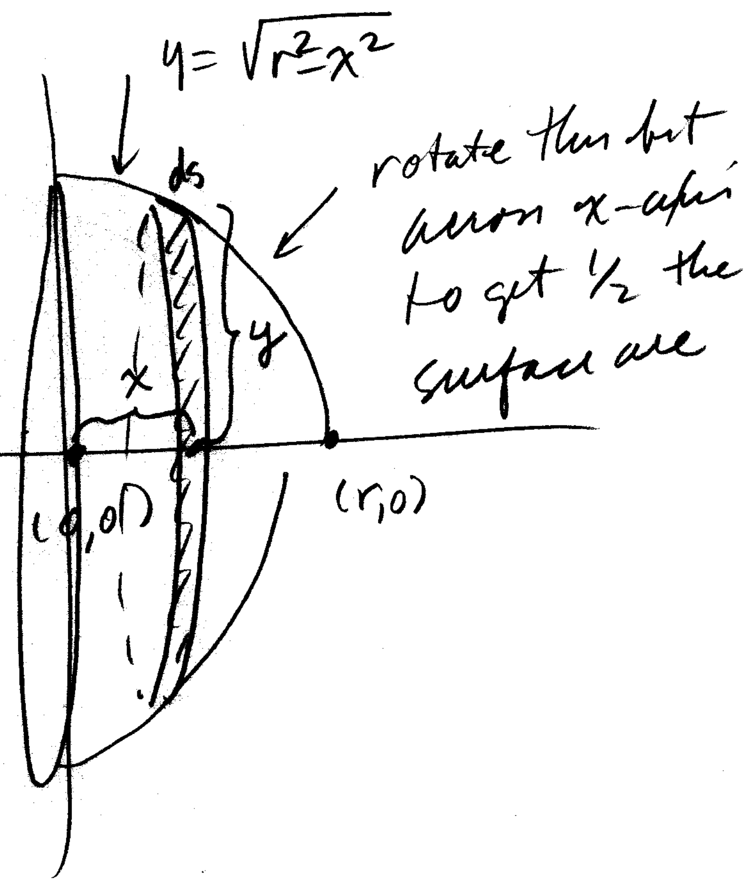
Hence,

$$\lim_{T \to \infty} \left[-\frac{e^{-sT}}{1+s^2} (\cos T + s \sin T) \right] = 0.$$

(4)
5pts

$dA = 2\pi y \, ds$

$= \cancel{2\pi y \sqrt{1+(y')^2}\, dx}$



$y = (r^2-x^2)^{1/2}$

$y' = \frac{1}{2}(r^2-x^2)^{-1/2}(-2x)$

$y' = \frac{-x}{\sqrt{r^2-x^2}}$

$(y')^2 = \frac{x^2}{r^2-x^2}$

$1+(y')^2 = 1 + \frac{x^2}{r^2-x^2}$

$= \frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}$

$= \frac{r^2}{r^2-x^2}$

Hence,

$dA = 2\pi y \, ds$

$= 2\pi y \sqrt{1+(y')^2}\, dx$

$= 2\pi \sqrt{r^2-x^2} \sqrt{\frac{r^2}{r^2-x^2}}\, dx$

$= 2\pi \sqrt{r^2-x^2} \cdot \frac{r}{\sqrt{r^2-x^2}}\, dx$

$= 2\pi r \, dx$

$$\frac{1}{2}A = \int_0^r dA$$

$$= \int_0^r 2\pi r\,dx$$

$$= 2\pi r x \Big|_0^r$$

$$= 2\pi r^2$$

Hence,

$$A = 4\pi r^2.$$