Linear Versus Nonlinear

Math 45 — Linear Algebra

David Arnold

David-Arnold@Eureka.redwoods.cc.ca.us

Abstract

Linear transformations must map lines to lines (or points). This is not the case with non-linear transformations. this Matlab activity will encourage visual exploration of this concept. *Prerequisites: Matrix multiplication, rudimentary knowledge of the nullspace, dependent and independent vectors, bases, linear transformations, Matlab's element-wise operators.*

Table of Contents

title page

contents

previous page

next page

back

print doc

close

exit

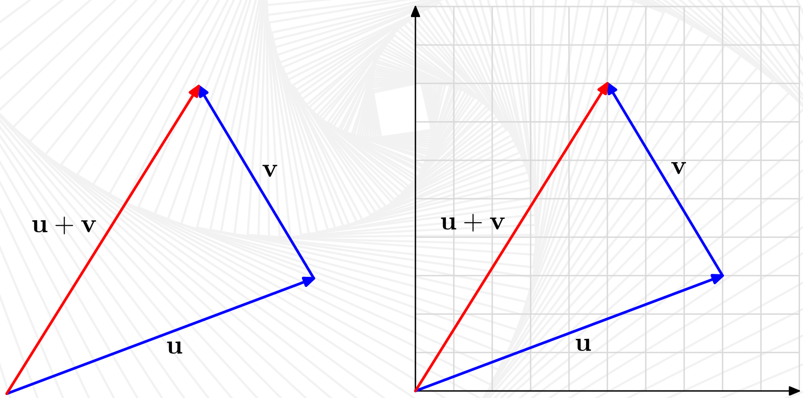
The Geometry of Vectors

We begin by establishing some fundamentals of the geometry of vectors.

Addition

First, let's look at how to add two vectors **u** and **v** without the use of a coordinate system. The standard procedure begins with drawing the vector **u**. Then we attach the vector **v** to the tip of the vector **u**, as shown in **Figure 1a**. Finally, the vector sum **u** + **v** is the vector drawn from the tail of vector **u** to the tip of vector **v**, also shown in **Figure 1a**.

If we place a coordinate system on the plane, then we can make an analytic connection to the practice of adding vectors.



(a) (b)

Figure 1 Adding vectors geometrically.

In **Figure 1b**, use the grid to note the horizontal and vertical displacements of the vectors **u** and **v**. It is easy to see that

$$\mathbf{u} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

Adding these vectors analytically,

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}8\\3\end{pmatrix}+\begin{pmatrix}-3\\5\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}.$$

This result says that the horizontal and vertical components of the vector sum $\mathbf{u}+\mathbf{v}$ are 5 and 8, respectively. It is easily checked in **Figure 1b** that these are the horizontal and vertical displacements of the vector $\mathbf{u}+\mathbf{v}$.

Scalar Multiplication

Next, what happens when we multiply a vector by a scalar? The simple answer is the fact that the vector is *scaled*. For example, if the vector $\mathbf{u}$ in **Figure 2a** is multiplied by 2, this produces the vector $2\mathbf{u}$ (pictured in **Figure 2b**) that points in the same direction as $\mathbf{u}$ (parallel to $\mathbf{u}$), but is twice as long. If the vector $\mathbf{u}$ is multiplied by -1.5, then the vector $-1.5\mathbf{u}$ (also pictured in **Figure 2b**) points in the *opposite* direction of $\mathbf{u}$ and its length (magnitude) is 1.5 times that of $\mathbf{u}$.

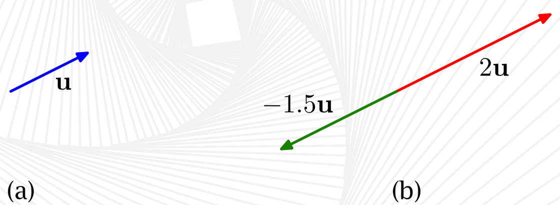


Figure 2 Multiplying by a scalar.

Again, if we impose a coordinate system, then we can make an analytic connection.

In **Figure 3a**, use the grid to note the horizontal and vertical displacements of the vector $\mathbf{u}$. It is easy to see that

$$\mathbf{u}=\begin{pmatrix}4\\2\end{pmatrix}.$$

If one multiplies vector $\mathbf{u}$ by 2 analytically, then

title page
contents
previous page
next page
back
print doc
close
exit

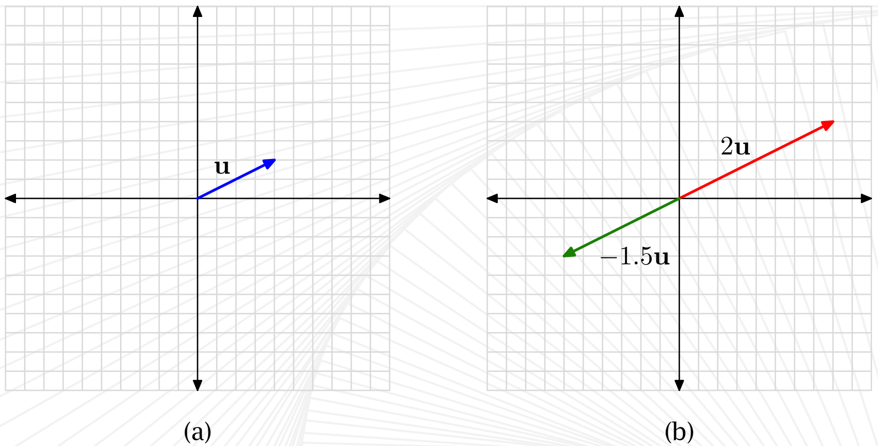


Figure 3 Multiplying by a scalar.

$$2\mathbf{u} = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}.$$

This result says that the horizontal and vertical displacements of the vector $2\mathbf{u}$ are 8 and 4, respectively. It is easily checked in **Figure 3b** that these are the horizontal and vertical displacements of the vector $2\mathbf{u}$. Similar comments are in order for the vector $-1.5\mathbf{u}$.

Subtraction

Subtraction of vectors is defined in the usual manner. That is, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. This idea can be used to subtract the vectors $\mathbf{u}$ and $\mathbf{v}$ in **Figure 4a**.

First, $-\mathbf{v}$ has the same magnitude as $\mathbf{v}$, but points in the *opposite* direction. Because $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, we add the vectors $\mathbf{u}$ and $-\mathbf{v}$ by attaching the tail of the vector $-\mathbf{v}$ to the tip of vector $\mathbf{u}$. The vector $\mathbf{u} - \mathbf{v}$, or $\mathbf{u} + (-\mathbf{v})$, is then drawn from the tail or beginning of vector $\mathbf{u}$ to the tip of vector $-\mathbf{v}$, as shown in **Figure 4b**.

However, in practice, one draws the difference $\mathbf{u} - \mathbf{v}$ simply by drawing a vector from the tip of vector $\mathbf{v}$ to the tip of vector $\mathbf{u}$, as shown in **Figure 5**.

title page
contents
previous page
next page
back
print doc
close
exit

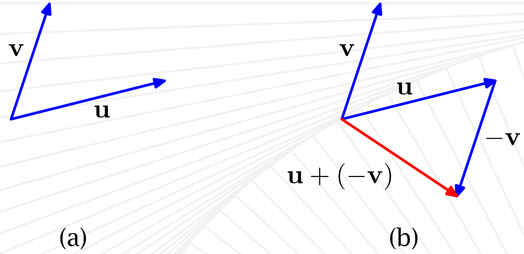


Figure 4 Subtracting vectors geometrically.

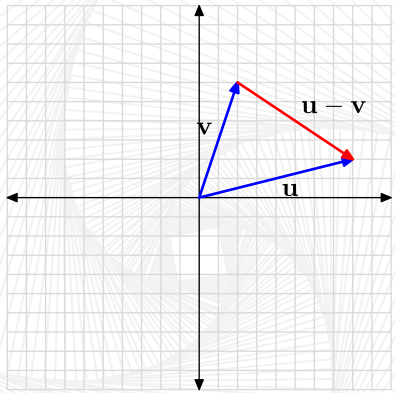


Figure 5 Subtracting vectors geometrically.

Note that the vector $\mathbf{u} - \mathbf{v}$ points in the same direction (parallel) as the vector $\mathbf{u} + (-\mathbf{v})$ in **Figure 4b** and has the same length. Consequently, it is the same vector. It is also important to note that the vector $\mathbf{u} - \mathbf{v}$ points toward the vector $\mathbf{u}$.

With the coordinate system imposed in **Figure 5**, one can make an analytic connection. It is not difficult to capture the horizontal and vertical displacements of the vectors $\mathbf{u}$ and $\mathbf{v}$ from the grid. Indeed,

$$\mathbf{u} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

Performing the subtraction analytically leads to the result

title page
contents
previous page
next page
back
print doc
close
exit

$$\mathbf{u} - \mathbf{v} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}.$$

Note that the horizontal and vertical displacements of the vector $\mathbf{u} - \mathbf{v}$ in **Figure 5** are 6 and -4, respectively.

Lines

We are all familiar with the equation of a line in the plane. For example, the line passing through the point $(2,3)$ with slope $m = -3/2$ is pictured in **Figure 6a**.

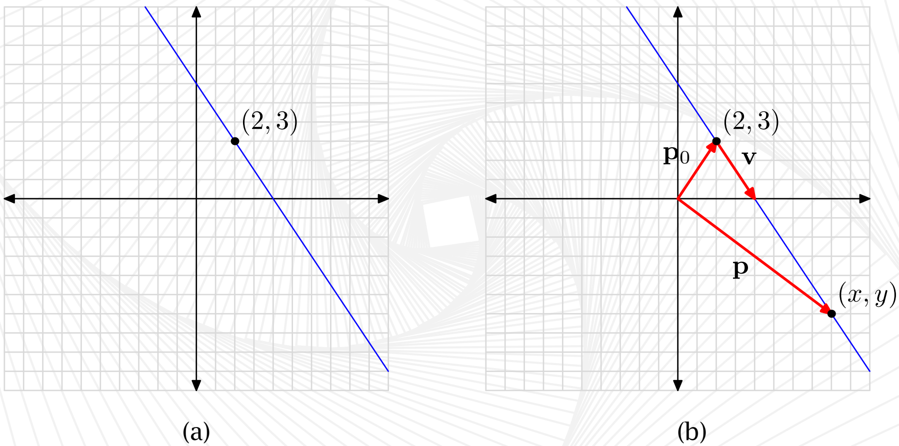


Figure 6 Finding the equation of a line.

We learned long ago that the equation of such a line is given by the formula

$$y - y_0 = m(x - x_0),$$

where m is the slope of the line and (x_0, y_0) are the coordinates of a given point on the line. In the case of the line in **Figure 6a**, the equation is

$$y - 3 = -\frac{3}{2}(x - 2),$$

title page
contents
previous page
next page
back
print doc
close
exit

or, after some simplification, $y = -(3/2)x + 6$. Unfortunately, this form of the line is not very useful in higher dimensions. For example, in R^3, we cannot speak well about the "slope" of a line, but we can talk about the "direction" of a line. Indeed, one need only find a *direction vector*, that is, a vector that points in the direction of the line. In **Figure 6b**, $\mathbf{v}$ is a direction vector for the line. We've also drawn two other vectors:

- The vector $\mathbf{p}_0$ is a vector that is drawn from the origin of the coordinate system to the given point (x_0, y_0).
- The vector $\mathbf{p}$ is a vector that is drawn from the origin to an arbitrary point (x, y) on the line.

It is important to note that this same description can be done in higher dimensions. For example, in R^3, we could select a point and a direction vector for the line, then craft a picture identical to that in **Figure 6b**. The only difference would be the extra dimension. With a little imagination, the same picture could be drawn in higher dimensions, but we'll leave that topic for another day.

The vector diagram in **Figure 6b** holds the key to finding the equation of the line in vector form. The idea is simplicity itself. Note that the vector $\mathbf{p} - \mathbf{p}_0$ is a vector starting from the tip of $\mathbf{p}_0$ and ending at the tip of the vector $\mathbf{p}$ (see **Figure 5**). Therefore, the vector $\mathbf{p} - \mathbf{p}_0$ is parallel to the direction vector and must be a scalar multiple of the vector $\mathbf{v}$. Thus, the equation of the line in **Figure 6b** is

$$\mathbf{p} - \mathbf{p}_0 = t\mathbf{v},$$

or, equivalently,

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v},$$

where t is an arbitrary scalar (in this case a real number). The bold of heart will recognize the striking similarity of the forms $y = b + mx$ and $\mathbf{p} = \mathbf{p}_0 + t\mathbf{v}$, but an example should help convince us that we have the correct equation of the line in vector form.

Example 1

Use Matlab to sketch the equation of the line passing through the point $(2, 3)$ with slope $-3/2$.

Solution. In **Figure 6b**, note that the slope of the line is $-3/2$. It is easy to use the slope to find a direction vector for the line. Using the concept of "rise over run," the vector $\mathbf{v}$ in **Figure 6b** is crafted by starting

title page
contents
previous page
next page
back
print doc
close
exit

at the point $(2,3)$, then moving to the right 2 units and down 3 units. Thus, a direction vector (there are others) for the line is

$$\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Further, a position vector, drawn to the given point, is

$$\mathbf{p}_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Consequently, the equation of the line, in vector form, is

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

where t is an arbitrary scalar. It is easy to see that

$$x = 2 + 2t,$$
$$y = 3 - 3t.$$

Matlab can easily plot these equations. Choose some values for the paramter t, then calculate both x and y.

```
t=linspace(-10,10);
x=2+2*t;
y=3-3*t;
```

Plot the graph with the following command.

```
plot(x,y)
```

This gives a nice graph, but we want to go a bit further so that our final result matches that in **Figure 6b**. First, restrict the axes and provide a grid.

```
axes([-10,10,-10,10])
grid
```

Next, we provide some decorative material to give our figure the professional touch. Actually, with the release of Matlab 5.3, you perform each of the following tasks interactively in the figure window. However, we want to make our readers aware that you also can do this from the command line.

```
set(gca,'xtick',-10:2:10,'ytick',-10:2:10);
line(2,3,'Marker','.','MarkerSize',16,'color','r')
text(2,3,'  (2,3)','HorizontalAlignment','Left')
```

The first coomand sets the tick marks on each axis. It does this by accessing a numerical "handle" for the axes (`gca`). The vector `-10:2:10` contains the numbers from −10 to 10, incremented by twos. The second line plots a single dot at the point $(2, 3)$ and sets marker type, size, and color. The third line typesets a label and sets its alignment (left, center, right). The effect of each of these commands is seen in **Figure 7**.

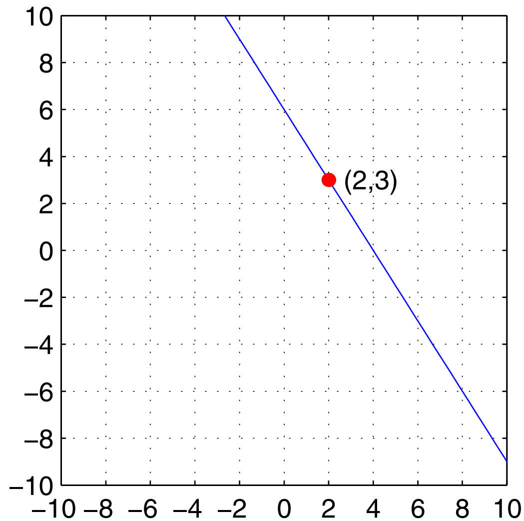


Figure 7 Drawing a line in Matlab.

title page
contents
previous page
next page
back
print doc
close
exit

When working with a large number of commands, it is more efficient to save your commands in a "script file." Open the Matlab editor with the command `edit`, then enter the commands you want executed in sequence. To produce the image in **Figure 7**, we saved the following code as `example1.m`.

```
close all
t=linspace(-10,10);
x=2+2*t; y=3-3*t; plot(x,y)
axis([-10,10,-10,10])
set(gca,'xtick',-10:2:10,'ytick',-10:2:10);
grid
line(2,3,'Marker','.','MarkerSize',16,'color','r')
text(2,3,'  (2,3)','HorizontalAlignment','Left')
set(gcf,'PaperPosition',[0,0,3,3])
print -depsc2 example1.eps
```

The second to last line sets an appropriate size (3 inches by 3 inches) so that the figure will fit nicely in a printed document. The last line saves the figure as an encapsulated postscript (`eps`) file that can later be imported into a document. To produce the figure and save the image file, all one needs to do at this point is to enter the name of the script file at the Matlab prompt.

```
>> example1
```

Script files are the most efficient way of interacting with Matlab. Should you make a mistake in a long sequence of commands executed at the Matlab prompt, it is frustrating and time consuming to begin such a sequence anew. However, if you are putting your commands into a script file, you need only adjust the file, save, then execute the name of the file at the Matlab prompt. Always work with script files when you can.

Linear Transformations

A *linear transformation* is a special type of function that maps one vector space to another. With that said, let's present the formal definition.

title page
contents
previous page
next page
back
print doc
close
exit

Definition 1

Let V and W be vector spaces. Let L be a function that maps the vector space V into the vector space W; i.e., $L : V \mapsto W$. L is a liner transformation if and only if

- $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V, and
- $L(\alpha\mathbf{u}) = \alpha L(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars α.

In this activity, we assume that you have some rudimentary experience with linear transformations, but we still want to review the ideas important to the successful completion of this exercise.

Example 2

Let A be an $m \times n$ matrix. Define $L : R^n \mapsto R^m$ by $L(\mathbf{x}) = A\mathbf{x}$. Show that L is a linear transformation.

Solution. We have to show that L satisfies both properties of Definition 1. First,

$$L(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = L(\mathbf{u}) + L(\mathbf{v}),$$

and secondly,

$$L(\alpha\mathbf{u}) = A(\alpha\mathbf{u}) = \alpha(A\mathbf{u}) = \alpha L(\mathbf{u}).$$

Therefore, L is a linear transformation.

Thus, all matrix mappings are linear transformations. The converse is also true. Any linear transformation from R^n to R^m can be represented by matrix multiplication. It can be shown that any linear transformation from R^n into R^m is completely determined by the action of the transformation on the so-called “standard basis” elements for R^n. This is somewhat intuitve due to the fact that every element in R^n can be written as a linear combination of its basis elements. If we know what the transformation does to the basis elements, linearity allows us to see what happens to any element. For example, suppose

title page
contents
previous page
next page
back
print doc
close
exit

$\mathbf{x}$ is an element of R^n and $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for R^n.[1] Then $\mathbf{x}$ can be written as a linear combination of the basis elements.

$$\mathbf{x} = x_1\mathbf{e}_2 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Next, the linearity of the transformation allows the following computation.

$$\begin{aligned} L(\mathbf{x}) &= L(x_1\mathbf{e}_2 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1L(\mathbf{e}_1) + x_2L(\mathbf{e}_2) + \cdots + x_nL(\mathbf{e}_n) \end{aligned}$$

Therefore, if the action of L is known on each basis element, the action of L on any element in R^n is known. Indeed, we can write

$$L(\mathbf{x}) = [L(\mathbf{e}_1),\ L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

which in matrix form, becomes

$$L(\mathbf{x}) = A\mathbf{x},$$

where

$$A = [L(\mathbf{e}_1),\ L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)] \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Therefore, the columns of A are the transformations of the columns of the identity matrix.

Example 3

Consider the transformation $L : R^2 \mapsto R^2$ defined by

[1] The standard basis for R^n is simply the collection of the column vectors of the identity matrix; $\mathbf{e}_1$ represents the first column of the identity matrix, $\mathbf{e}_2$ represents the second column, and so on.

title page
contents
previous page
next page
back
print doc
close
exit

$$L(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}. \qquad (1)$$

Show that this transformation is linear and can be represented by matrix multiplication.

Solution. The proof that this transformation is linear is left to the exercises. We find only the matrix of the transformation. First, compute what the linear transformation does to each of the standard basis elements. This is somewhat obvious if you examine the geometry of the transformation. The point (x_1, x_2) is mapped to the point (x_2, x_1). This is a reflection across the line $x_1 = x_2$, as shown in **Figure 8**.

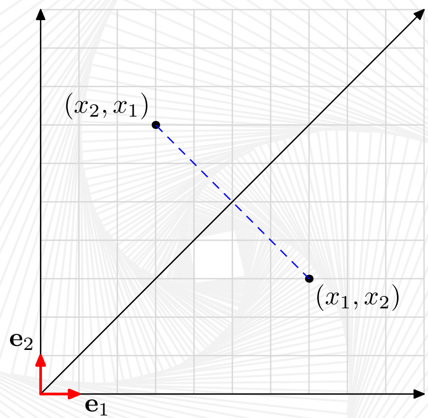


Figure 8 Reflecting across the line $x_1 = x_2$.

It is easy to see that the transformation "reflects" $\mathbf{e}_1$ onto $\mathbf{e}_2$ and vice-versa. However, this can also be shown analytically using **equation 1**.

$$L(\mathbf{e}_1) = L(1, 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}2$$

$$L(\mathbf{e}_2) = L(0, 1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}1$$

Therefore, the matrix of the transformation is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

That this is correct is easily checked.

$$L(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

Mapping Lines and Points

For the remainder of the activity, we will concentrate on linear transformations that map R^2 to R^2, though all of what we say applies equally well to higher dimensions. In this case, any linear transformation $L : R^2 \mapsto R^2$ can be represented by matrix multiplication. That is, if $L : R^2 \mapsto R^2$ is a linear transformation, then there exists a matrix A such that

$$L(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x}$ in R^2.

It is completely obvious that a linear transformation must take a point to another point, but what happens when a linear transformation is applied to a line? Certainly, in some cases it is obvious that the linear transformation will take lines to lines. For example, the linear transformation that reflects points across the horizontal axis will clearly map lines to lines, as will the transformation that rotates points about the origin by a fixed angle. But what can happen with an arbitrary linear transformation?

Example 4

Use Matlab to show that the linear transformation $L(\mathbf{x}) = A\mathbf{x}$, where A is the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

appears to map lines to lines.

Solution. First, choose a line in R^2. For example, if $\mathbf{p}_0 = (1, -1)$ is a point on the line and $\mathbf{v} = (2, -1)$ is a direction vector for the line, then the equation of the line is

title page
contents
previous page
next page
back
print doc
close
exit

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Thus,

$$x = 1 + 2t,$$
$$y = -1 - t,$$

and we can plot the line satisfying these relations as we did in **Example 1**. First, select values for the parameter t and calculate x and y.

```
t=-10:10;
x=1+2*t;
y=-1-t;
```

The command

```
plot(x,y,'b.')
```

will produce a set of blue, discrete points, similar to that shown in **Figure 9a**.

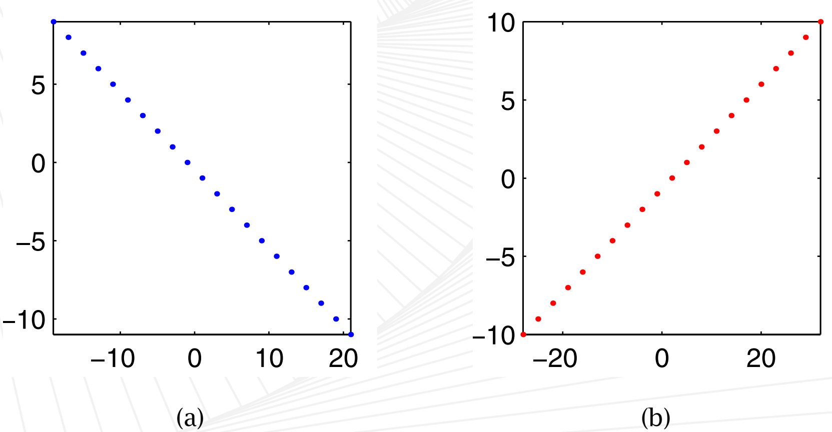


Figure 9 Linear transfromation of a line.

title page
contents
previous page
next page
back
print doc
close
exit

Next, we load the matrix that defines our transformation.

```
A=[1,-1;1,1];
```

We could multiply the coordinates of each blue point in **Figure 9a** by matrix A, but it is much more efficient to transform all of the points at once. With this in mind, we create an input matrix that contains the blue points of **Figure 9a** as columns.

```
Input=[x;y];
```

It is instructive to remove the semicolon the suppresses the output of this command and study the contents of the matrix `Input`. Next, we transform each point by hitting each point in the input matrix with the matrix A. This is easily accomplished with the following command.

```
Output=A*Input;
```

Again, remove the suppressing semicolon and note that each column of the matrix `Output` contains the transform of the corresponding column of the `Input` matrix. If we let (u, v) represent the coordinates of the transformed point, then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$u = x - y,$$
$$v = x + y.$$

Consequently, the first row of the `Output` matrix contains the u-values of the transformed points and the second row contains the v-values. We strip these off with these commands.

```
u=Output(1,:);
v=Output(2,:);
```

There are several plotting options available. Let's open a second figure window with this command.[2]

[2]Matlab will by default plot in the figure window that was last active. Be sure to go directly to the command line if you wish your plot to appear in this new window.

title page
contents
previous page
next page
back
print doc
close
exit

```
figure
```

We don't yet know that a linear relationship exists between the transformed points, so it would be presumptious on our part to connect the discrete points with line segments. Thus, we will plot the transformed points as discrete points and hope that we can see a pattern in the plot. The command

```
plot(u,v,'r.')
```

was used to plot the image in **Figure 9b**. The relationship appears linear but our experiment does not constitute a proof that the relationship is actually linear.

Take a moment to replay the commands of this example with a smaller increment for the time interval. Try these values for the parameter t:

```
t=-10:0.5:10
```

and

```
t=-10:0.1:10
```

The transformation of **Example 4** appears to map lines to lines. But is this always the case?

Example 5

Consider the linear transformation $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Use Matlab to investigate what happens to various lines under this transformation.

Solution. We will examine what happens to three different lines under this transformation. The first two lines to be transformed are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

title page
contents
previous page
next page
back
print doc
close
exit

while the third line has equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Following the lead of **Example 4**, we plot the lines as follows.

```
t=-10:10;
x1=1-t;
y1=3+t;
x2=-2+t;
y2=-1+t;
x3=1-2*t;
y3=-3+t;
plot(x1,y1,'r.',x2,y2,'g+',x3,y3,'bo')
```

These lines are shown in **Figure 10a**.

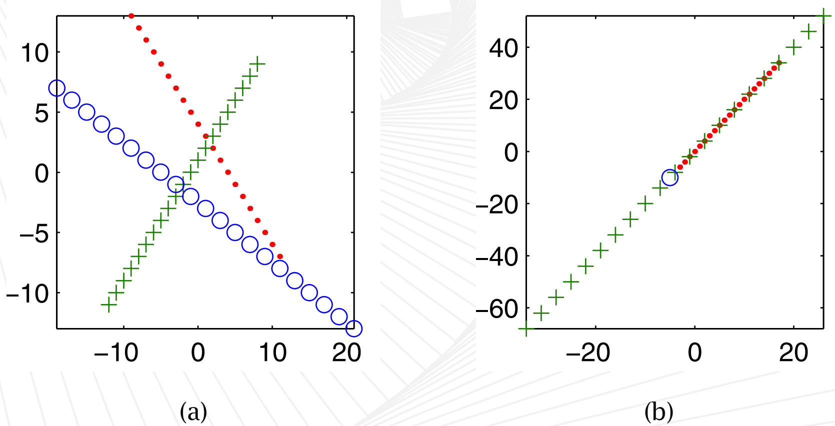


Figure 10 Linear transformation of a line.

Note that we used a different linestyle, as well as a different color, for each line. Enter the matrix A

title page
contents
previous page
next page
back
print doc
close
exit

```
A=[1,2;2,4];
```

and build the input matrices for each line.

```
Input1=[x1;y1];
Input2=[x2;y2];
Input3=[x3;y3];
```

Again, it is instructive to remove the semicolons and note that each column of each input matrix contains one point on the corresponding line. Next, hit each input matrix with the transformation matrix.

```
Output1=A*Input1;
Output2=A*Input2;
Output3=A*Input3;
```

Strip u-values from the first row of the output matrix, the v-values from the second row.

```
u1=Output1(1,:);
v1=Output1(2,:);
u2=Output2(1,:);
v2=Output2(2,:);
u3=Output3(1,:);
v3=Output3(2,:);
```

Plot each of the transformed lines in a new figure window. These commands should give you an image similar to that in **Figure 10b**.

```
figure
plot(u1,v1,'r.',u2,v2,'g+',u3,v3,'bo')
```

Note that the transformation of the red dots on the first line appear to land on a line in **Figure 10b**, as do the green plus signs of the second line. However, such is not the case with the blue circles of the third line. They are all transformed onto a single point! Why is that?

The simple answer is the fact that the vector $(-2, 1)$ lies in the nullspace of the transformation matrix (check this). Thus,

title page
contents
previous page
next page
back
print doc
close
exit

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

becomes

$$\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \left[\begin{pmatrix} 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right], \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \left[t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right], \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + t \left[\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right], \\
&= \begin{pmatrix} -5 \\ -10 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&= \begin{pmatrix} -5 \\ -10 \end{pmatrix}.
\end{aligned}$$

Every point on the blue line maps onto the point $(-5, -10)$! Try zooming in the figure window to verify this fact.

Providing a Proof

In **Example 4**, we saw that lines were transformed to lines, but in **Example 5** we saw two possibilities. Some lines were taken to lines, but others were transformed into single points. Are there any other possibilities?

The short answer is no, but it is now time to provide a proof of this fact. The proof is remarkable, for it is as simple as it is short.

Proposition 1

Suppose that L is a linear transformation that maps R^2 into R^2. Suppose that $\mathbf{p} = \mathbf{p}_0 + t\mathbf{v}$ is a line in R^2. Then the image of this line under the transformation L is either a line or a point. There are no other possibilities.

title page
contents
previous page
next page
back
print doc
close
exit

Proof. Remember, if you see a vector equation having the form $\mathbf{p} = \mathbf{p}_0 + t\mathbf{v}$, then the set of points satisfying this equation forms a line in R^2 (and higher dimensional spaces). The transform of this line is

$$A\mathbf{p} = A(\mathbf{p}_0 + t\mathbf{v}),$$
$$A\mathbf{p} = A\mathbf{p}_0 + A(t\mathbf{v}),$$
$$A\mathbf{p} = A\mathbf{p}_0 + t(A\mathbf{v}).$$

If $\mathbf{v}$ is in the nullspace of A, then $A\mathbf{v} = \mathbf{0}$ and the transform of the line is the single point $A\mathbf{p}_0$. However, if $\mathbf{v}$ is not in the nullspace of A, then the transform has the form $A\mathbf{p} = A\mathbf{p}_0 + t(A\mathbf{v})$, which is the vector form of a line. The proposition is proved!

Linear Versus Nonlinear

If the columns of A are independent, then the nullpace of A contains only the zero vector. In this case, the transformation $L(\mathbf{x}) = A\mathbf{x}$ cannot map every point on a line to a single point. We can show that points on a line segment joining two points in the plane are sent to the line segment joining the images of these points. This is tremendously useful, because now we can look at the transformations of polygons, but that's another story.

Let's end our activity by contrasting linear and nonlinear transformations. Some simple examples will make our final point, but first let's examine Matlab's `meshgrid` command. The command

```
[X,Y]=meshgrid(-2:2);
```

feeds the vector `[-2,-1,0,1,2]` to the `meshgrid` command. The `meshgrid` command is best explained by examining the output of the command.

```
X =

    -2    -1     0     1     2
    -2    -1     0     1     2
    -2    -1     0     1     2
    -2    -1     0     1     2
    -2    -1     0     1     2
```

```
Y =

    -2    -2    -2    -2    -2
    -1    -1    -1    -1    -1
     0     0     0     0     0
     1     1     1     1     1
     2     2     2     2     2
```

You can best understand the `meshgrid` command by visualizing a grid of points having coordinates (x, y), where x is a value chosen from the matrix X and y is a value chosen from the matrix Y. This is most easily imagined by superimposing the matrix Y onto the matrix X, creating a grid of coordinates having the form shown in **Figure 11**.

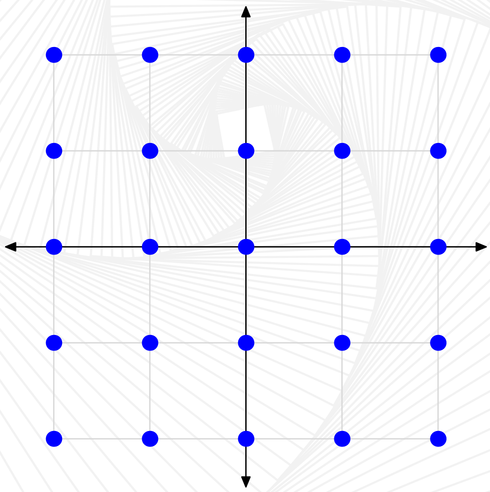
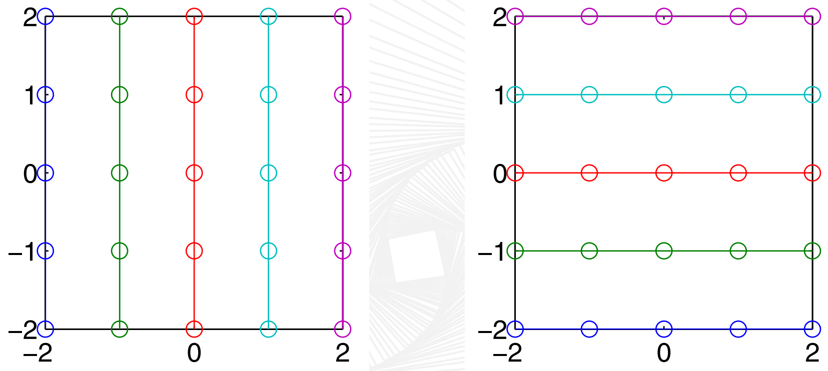


Figure 11 Visualizing the `meshgrid` command.

Matlab's `plot` command is an extremely flexible tool. We've used it to plot one vector against another, but it also can be used to plot a matrix versus vector, or a matrix versus a matrix. In our case, we wish to plot the matrix Y versus the matrix X. What will happen when we execute the command `plot(X,Y)`? The command

```
plot(X,Y)
```

generates the image shown in **Figure 12a**. What has happened? Matrix X and matrix Y each have 5 columns. The command `plot(X,Y)` first plots the first column of matrix Y versus the first column of matrix X in blue. Then the second column of Y is plotted versus the second column of X in green, and so on until all the columns are exhausted.



(a) (b)

Figure 12 Plotting a matrix versus a matrix.

The command

```
plot(X',Y')
```

will plot the rows of Y versus the rows of X, as shown in **Figure 12b**.

Armed with this knowledge of the `meshgrid` and `plot` commands, let's investigate how linear and nonlinear transforms treat lines.

Example 6

Consider the linear transformation

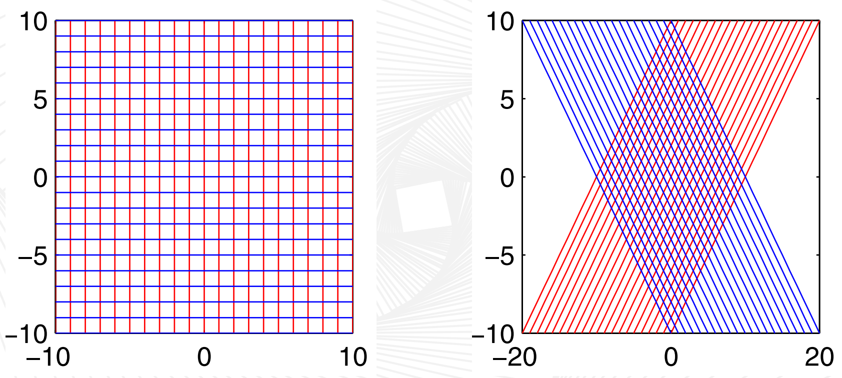
title page
contents
previous page
next page
back
print doc
close
exit

$$L(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

Use Matlab to transform a grid of lines.

Solution. First, set up a grid of lines. The following commands were used to produce the grid of lines in **Figure 13a**. Note that the horizontal lines are colored blue, the vertical red.

```
[X,Y]=meshgrid(-10:10);
plot(X,Y,'r',X',Y','b')
```



(a) (b)

Figure 13 Transforming a grid of lines.

If $L : (x, y) \mapsto (u, v)$, then $L(\mathbf{x}) = A\mathbf{x}$ becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

so

$$u = x + y,$$
$$v = -x.$$

One way to proceed would be to write some sort of loop. In the loop, each point (x, y) would be transformed into (u, v) coordinates. However, this is both time consuming and overly complicated. After all, Matlab is a matrix orientated environment, so we can transform all of the points in the grid with these commands.

```
U=X+Y;
V=-X;
```

Create a new figure window and use the `plot` command to plot the transformed lines. These commands should produce an image similar to that in **Figure 13b**.

```
figure
plot(U,V,'r',U',V','b')
```

It is important to understand the point being made here, and that is the fact that the linear transformation sends lines to lines.

By this time, you have probably guessed what a nonlinear transformation will do (or won't do).

Example 7

Consider the transformation

$$L(x, y) = \begin{pmatrix} x^2 + y \\ x + y \end{pmatrix}.$$

Use the technique developed in **Example 6** to examine the images of lines under this transformation.

Solution. First, note that the transformation $L(x, y) = (x^2 + y, x + y)$ is nonlinear. We will leave it as an exercise for you to prove this fact.

Next, the following commands were used to create the grid of lines shown in **Figure 14a**. Horizontal lines are blue, vertical are red.

```
[X,Y]=meshgrid(-10:10);
plot(X,Y,'r',X',Y','b')
```


title page
contents
previous page
next page
back
print doc
close
exit

If $L : (x, y) \mapsto (u, v)$, then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x^2 + y \\ x + y \end{pmatrix},$$

so

$$u = x^2 + y,$$
$$v = x + y.$$

Again, we use Matlab's matrix capability to make this transformation on the entire grid. However, you must remember that to square each element of the matrix X, you must use Matlab's elementwise operater, `.^`.

```
U=X.^2+Y;
V=X+Y;
```

Prepare a new figure window, then plot the transform of each line shown in **Figure 14a**. The following commands were used to prepare the image in **Figure 14b**.

```
figure
plot(U,Y,'r',U',V','b')
```

The main point of this activity should now be clear. Nonlinear transformations don't necessarily take lines to lines. It is interesting to note in this case that the vertical lines were transformed into lines, but the horizontal lines were transformed into parabolas (can you prove this?).

Homework

Consider the following vectors from R^2.

$$\mathbf{u} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

In each of the following exercises, find the indicated sum, difference, or scalar product

title page
contents
previous page
next page
back
print doc
close
exit

- analytically, and
- by graphing on a sheet of graph paper.

title page
contents
previous page
next page
back
print doc
close
exit

1. $\mathbf{u}+\mathbf{v}$
2. $\mathbf{u}+\mathbf{w}$
3. $\mathbf{v}-\mathbf{w}$
4. $\mathbf{u}-\mathbf{w}$
5. $-3\mathbf{u}$ and $2\mathbf{w}$
6. $-2.5\mathbf{u}$ and $3\mathbf{v}$

Use Matlab to sketch the graph of each of the lines given in vector form.

7. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

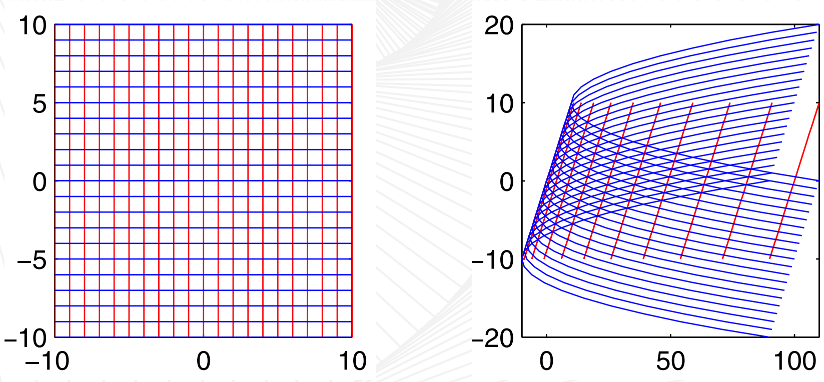
8. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

9. Prove that the transformation defined in **Example 3** is linear.

Show that each of the transformations can be written as a matrix product by finding a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for the given transformation.

10. $L(x,y) = \begin{pmatrix} x+2y \\ 3x-y \end{pmatrix}$

11. $L(x,y,z) = \begin{pmatrix} x \\ x-y \\ x+y+2z \end{pmatrix}$



(a) (b)

Figure 14 Transforming a grid of lines.

title page
contents
previous page
next page
back
print doc
close
exit

Use the technique of **Example 4** to show that the given transformation maps the given line to a line in R^2. Provide a both plot of the line and its image under the transformation. You might want to research the `subplot` command to save on paper.

12. $L(x, y) = \begin{pmatrix} x + y \\ -y \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
13. $L(x, y) = \begin{pmatrix} x - y \\ 2x + y \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

14. In **Example 5**, show how you can use block multiplication to consolidate

```
Output1=A*Input1;
Output2=A*Input2;
Output3=A*Input3;
```

    into a command involving only one multiplication.
15. Prove that the transformation of **Example 7** is nonlinear.

Use the technique of **Example 6** and **Example 7** to map a grid of lines to their transforms in each of the following exercises.

16. $L(x, y) = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}$
17. $L(x, y) = \begin{pmatrix} x + y^2 \\ y - x^2 \end{pmatrix}$

title page

contents

previous page

next page

back

print doc

close

exit