Graphs and Matlab

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Abstract

Graphs are introduced along with their adjacency matrix. Paths are defined and an algorithm posed for counting paths of length n.

Prerequisites. Matrix multiplication. A little knowledge of the transpose of a matrix. You will also need to download and use the Matlab file `graph.m`.

1 Graphs

A *graph* is a finite collection of *vertices* or *nodes* together with a set of edges which join certain pairs of vertices. Think of the edges of a graph as two-way streets along which traffic can flow in either direction.

The graph shown in Figure 1 has six vertices or nodes: P_1, P_2, P_3, P_4, P_5, and P_6.

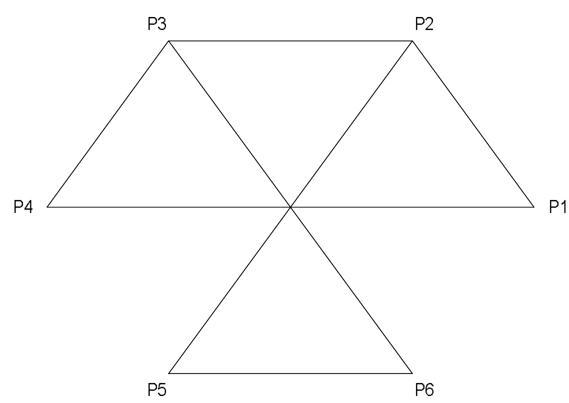


Figure 1. An undirected graph.

The *edges* of the graph in Figure 1 are the segments connecting P_1 and P_2, P_1 and P_4, P_2 and P_3, P_2 and P_5, P_3 and P_4, and P_5 and P_6.

1.1 The Adjacency Matrix

Associated with each graph is an *adjacency* matrix $A = (a_{ij})$ which is constructed according to the following rule[1].

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge connecting } P_j \text{ to } P_i. \\ 0 & \text{otherwise} \end{cases}$$

The graph in Figure 1 has six vertices: P_1, P_2, P_3, P_4, P_5, and P_6. Therefore, the adjacency matrix will have six rows and six columns. Create a blank template for the adjacency matrix. Label the columns of the matrix with the names of the vertices. Similarly, label the rows of the matrix with the names of the vertices, *but place these on the right side of the matrix.*

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
						P_1
						P_2
						P_3
						P_4
						P_5
						P_6

In Figure 1 there is an edge connecting P_2 to P_5, so $a_{52} = 1$. This same edge connects P_5 to P_2, so $a_{25} = 1$. There is *no* edge connecting P_3 to P_1, so $a_{13} = 0$. There is no edge connecting P_6 to P_6, so $a_{66} = 0$.

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
		0				P_1
				1		P_2
						P_3
						P_4
	1					P_5
					0	P_6

Use the graph in Figure 1 to fill in the remainder of the adjacency matrix as follows.

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
0	1	0	1	0	0	P_1
1	0	1	0	1	0	P_2
0	1	0	1	0	1	P_3
1	0	1	0	0	0	P_4
0	1	0	0	0	1	P_5
0	0	1	0	1	0	P_6

[1] The notation a_{ij} represents the element that is located in the ith row and jth column of the matrix A.

Therefore, the adjacency matrix for the digraph in Figure 2 is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Note that the matrix A is *symmetric* with respect to its main diagonal. This is so because an edge connecting P_i to P_j implies that there is also an edge connecting P_j to P_i. You can easily check for symmetry by taking the transpose. If $A^T = A$, then the matrix A has to be symmetric[2].

Enter the matrix A.

```
>> A=[0 1 0 1 0 0;1 0 1 0 1 0;0 1 0 1 0 1;1 0 1 0 0 0;0 1 0 0 0 1;0 0 1 0 1 0]

A =

     0     1     0     1     0     0
     1     0     1     0     1     0
     0     1     0     1     0     1
     1     0     1     0     0     0
     0     1     0     0     0     1
     0     0     1     0     1     0
```

Take the transpose of the matrix A.

```
>> A'

ans =

     0     1     0     1     0     0
     1     0     1     0     1     0
     0     1     0     1     0     1
     1     0     1     0     0     0
     0     1     0     0     0     1
     0     0     1     0     1     0
```

Note that matrix A^T is identical to the matrix A. This is always the case when the matrix is symmetric with respect to its main diagonal.

1.2 Checking with `graph.m`

I have created a Matlab M-file called **graph.m**. If you provided the adjacency matrix as input, then the function **graph.m** will draw the associated graph[3]. I

[2]Linear algebra texts often use the symbol A^T to represent the transpose of matrix A. Matlab uses a single apostrophe to represent the transpose.

[3]You must place the file `graph.m` in a folder or directory on your matlab path.

decided to arrange the vertices on the unit circle to simplify the programming of `graph.m`. The first vertex is placed at $(1, 0)$ and the remaining vertices are spaced at equal increments along the unit circle in a counter-clockwise direction.

In practice, you can arrange your nodes or vertices in any geometric pattern you wish. In fact, if your vertices represent major airports in the United States, you might want to get a map of the country and place each node atop the city where the airport is located.

It is easy to use the file `graph.m` to draw a graph based on the adjacency matrix A.

```
>> graph(A)
```

2 Paths

A sequence of edges connecting vertex P_i to P_j is called a *path*. The length of a path is the number of edges used in the sequence.

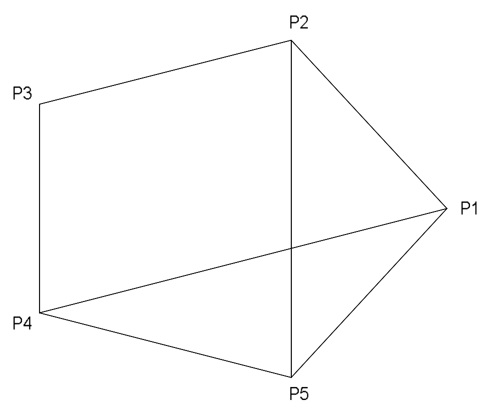


Figure 2.

In Figure 2, the single edge P_1P_2 connecting P_1 to P_2 is a path of length one. The path from P_1 to P_2 via the vertex at P_5 uses the edges P_1P_5 and P_5P_2 and has length two. The path from P_1 to P_2 via vertices P_4 and P_3 uses the edges P_1P_4, P_4P_3, and P_3P_2 and has length 3.

A path is called a *simple* path if no edge is traversed more than once.

2.1 Counting Paths of Length Two

The adjacency matrix for the graph in Figure 2 is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Use Matlab to compute A^2.

```
>> A=[0 1 0 1 1;1 0 1 0 1;0 1 0 1 0;1 0 1 0 1;1 1 0 1 0]

A =

     0     1     0     1     1
     1     0     1     0     1
     0     1     0     1     0
     1     0     1     0     1
     1     1     0     1     0

>> A^2

ans =

     3     1     2     1     2
     1     3     0     3     1
     2     0     2     0     2
     1     3     0     3     1
     2     1     2     1     3
```

Note that there are 2 paths of length two from P_1 to P_5:

- the path from P_1 to P_2 to P_5, and
- the path from P_1 to P_4 to P_5.

Note that the element fifth row, first column of A^2 is 2, the number of paths of length two from P_1 to P_5.

There are 3 paths of length two from P_2 to P_2:

- the path from P_2 to P_1 to P_2,
- the path from P_2 to P_3 to P_2, and
- the path from P_2 to P_5 to P_2.

The element in the second row, second column of A^2 is 3, the number of paths of length 2 from P_2 to P_2.

It is no coincidence that the number of paths of length two can be found be examining the square of the adjacency matrix. In general, if $a_{ij}^{(2)}$ is the element in the ith row and jth column of A^2, then $a_{ij}^{(2)}$ is the number of paths of length two from P_j to P_i.

2.2 Counting the Paths of Length N

In general, if $a_{ij}^{(N)}$ is the element in the ith row, jth column of matrix A^N, then $a_{ij}^{(N)}$ is the number of paths of length N from P_j to P_i.

For example, use Matlab to compute A^3.

```
>> A^3

ans =

     4     7     2     7     5
     7     2     6     2     7
     2     6     0     6     2
     7     2     6     2     7
     5     7     2     7     4
```

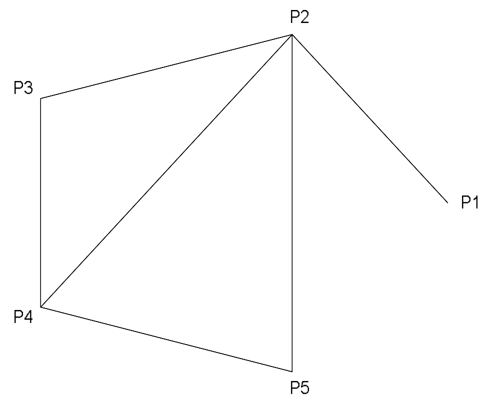
The element in the first row, fourth column of A^3 is 7. Therefore, there are 7 paths of length three from P_4 to P_1:

- $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$[4]
- $4 \rightarrow 3 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 1 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 5 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 5 \rightarrow 2 \rightarrow 1$
- There are two more paths of length three. Can you find them?

3 Homework

1. Create an adjacency matrix A for the following graph.

[4] Let the notation $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ represent the path from P_4 to P_3 to P_2 to P_1.



- Check your result with `graph.m`.
- Check that A is symmetric with respect to its main diagonal. Show that A and A^T are equal.

2. Draw the graph associated with the adjacency matrix

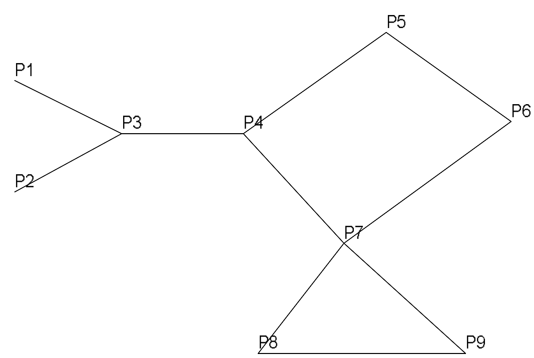
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

   Use the M-file `graph.m` to check your result.

3. What are the remaining two paths of length three in Figure 2 from P_4 to P_1?

4. Use Matlab to compute the number of paths of length four from P_2 to P_3 in Figure 2. List each of these paths.

5. Find the number of paths of length 9 from P_1 to P_9 in the following graph.



6. If A is an adjacency matrix for a graph and $R = A + A^2 + A^3$, then r_{ij} represents the number of paths from P_j to P_i of length 1, 2, or 3.

   (a) Let A be the adjacency matrix for the graph in Figure 2. Use Matlab to compute $R = A + A^2 + A^3$.

   (b) What is r_{25}?

   (c) The element r_{25} should represent the number of paths of length 1, 2, or 3 from P_5 to P_2.

      (a) List all paths of length 1 from P_5 to P_2.
      (b) List all paths of length 2 from P_5 to P_2.
      (c) List all paths of length 3 from P_5 to P_2.
      (d) Is the total number of paths of length 1, 2, or 3 from parts i, ii, and iii equal to r_{25}?