Pretest Exam #2
Linear Algebra
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Instructions. *Place the solution to each of the following questions on your own paper.*

1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 0 & 4 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}.$$

   a) Verify that $A(B+C) = AB + AC$.
   b) Verify that $(AB)^T = B^T A^T$.
   c) Verify that $(AB)^2 \neq A^2 B^2$.
   d) Find A^{-1}.
   e) Solve $AX + B = C$ for X.
   f) Find a matrix D such that $CD = 0$.
   g) Verify that $\det(AB) = \det(A)\det(B)$.
   h) Verify that $\det(A+C) \neq \det(A) + \det(C)$.
   i) Verify that $\det(A^T) = \det(A)$.

2. Prove: If A is nonsingular, than A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.
3. Prove: If A and B are nonsingular, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.
4. Let A be an $n \times n$ matrix.
   a) Prove: AA^T is a symmetric.
   b) Prove: $A - A^T$ is skew-symmetric.
5. Let A and B be symmetric $n \times n$ matrices. Prove: AB is symmetric if and only if $AB = BA$.
6. Consider the matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   a) State the inverse of each of the matrices E_1, E_2, and E_3.
   b) Clearly state what happens to a matrix A should it be multiplied on the left by each of the matrices E_1, E_2, and E_3.
   b) Clearly state what happens to a matrix A should it be multiplied on the right by each of the matrices E_1, E_2, and E_3.

7. Let A and B be $n \times n$ matrices. Prove: If B is singular, then AB is singular.
8. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & 2 \\ 2 & 0 & 5 \end{pmatrix}.$$

9. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ -x_1 - x_2 + 2x_3 &= -1\,. \\ 2x_1 + 5x_3 &= 4 \end{aligned}$$

   a) Write the system of equations in matrix form $A\mathbf{x} = \mathbf{b}$.
   b) Because A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, $\mathbf{x} = A^{-1}\mathbf{b}$. Use this fact and the inverse found in the previous problem to find the unique solution of the system.

10. Use elementary row operations to reduce

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 7 & 0 \\ -1 & -4 & 2 \end{pmatrix}$$

to row echelon form U. Use your work to factor A as a product of a lower and upper triangular matrix. That is, find L and U so that $A = LU$, or, in the language of linear algebra, perform an LU decomposition.

11. Evaluate the following determinant using cofactor expansion.

$$\begin{vmatrix} -1 & 2 & -3 \\ 2 & 4 & 6 \\ -1 & -1 & 4 \end{vmatrix}$$

12. Let A be an $n \times n$ matrix and let $\mathbf{b}$ a $n \times 1$ column vector. Partition A and $\mathbf{b}$ and use block multiplication to show

$$A\mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 + \cdots + b_n\mathbf{a}_n.$$

Use this structure to perform the following multiplication. Show your partition and all of your steps.

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 2 & 2 \\ 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

13. Let $\mathbf{r}$ be a $1 \times n$ row vector and let B be a $n \times n$ matrix. Partition $\mathbf{r}$ and B and use block multiplication to show that

$$\mathbf{r}B = r_1\mathbf{b}(1,:) + r_2\mathbf{b}(2,:) + \cdots + r_n\mathbf{b}(n,:).$$

Use this structure to perform the following multiplication. Show your partition and all of your steps.

$$\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 \\ 2 & 2 & 2 \\ 5 & 0 & -2 \end{pmatrix}$$

14. Perform the following block multiplications.

a) $A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$

b) $\begin{pmatrix} \mathbf{r}(1,:) \\ \mathbf{r}(2,:) \\ \vdots \\ \mathbf{r}(n,:) \end{pmatrix} B$

15. If X and Y are $n \times n$ matrices, show that

$$XY^T = \mathbf{x}_1\mathbf{y}_1^T + \mathbf{x}_2\mathbf{y}_2^T + \cdots + \mathbf{x}_n\mathbf{y}_n^T.$$

16. Use elementary row operations to place the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 5 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 3 & 7 & 0 & 0 \end{pmatrix}$$

in upper triangular form. Use this result to evaluate the determinant of A.

17. Suppose that A and B are matrices with $\det(A) = -3$ and $\det(B) = 5$. Evaluate each of the following.

a) $\det(3A)$ b) $\det(A^T)$ c) $\det(B^{-1})$ d) $\det(A^{-1}B)$

18. Let E_1, E_2, and E_3 be elementary matrices of Type I, Type II, and Type III, respectively. Let E_2 be the elementary matrix formed by multiplying its second row by 4. Further, suppose $\det(A) = -5$. Evaluate each of the following.

a) $\det(E_1A)$ b) $\det(E_2E_1A)$ c) $\det(E_1E_2E_3A)$ d) $\det(E_1E_2^{-1}A)$

19. Find the adjoint of the matrix

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 4 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix}.$$

a) Use the formula $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ to compute the inverse of the matrix A.

b) Find the inverse of matrix A by reducing A to I, then applying the same elementary row operations to the identity matrix to produce A^{-1}. Compare this result to the answer found in part (a).

20. Use Cramer's Rule to solve the following system of equations.

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ x_1 - 8x_2 &= 4 \end{aligned}$$

21. Use Cramer's Rule to solve the following system of equations.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 6 \\ x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 - 3x_3 &= 9 \end{aligned}$$