College of the Redwoods
Mathematics Department
Math 45—Linear Algebra

Exam #4
Linear Algebra
Matlab Component

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Essay Questions

Directions: *You must complete this exam in one sitting in Room 116 of the physical sciences building. You may take the exam Monday, Tuesday, or Wednesday, December 4–6. You must follow directions exactly to receive credit for your solutions. Please do not share any questions or results on this examination with fellow students until all have taken the examination. Finally, this examination is closed book, closed notes. You may not view any activities online to aid in the completion of this examination. You are free, however, to use any of Matlab's help files. Good luck!*

EXERCISE 1. Consider the following set of data.

t	y
0	159
5	1489
10	2099
15	1936
20	898

(a) Use the technique of least squares to fit a parabola having form $y = at^2 + bt + c$ to the data set. Show all preparatory work on college ruled paper. Your work should include the following:

- A system of equations generated by substituting each data point from the table into $y = at^2 + bt + c$.
- A matrix equation for the system.
- The solutions for a, b, and c and the equation $y = at^2 + bt + c$.
- Any other intermediate results you deem important.

(b) Plot the data set and the parabola of best fit on the same plot. Use the `xlabel`, `ylabel`, and `title` commands to provide appropriate axes labels and a title containing the equation of the parabola.

(c) Prepare an M-file containing all commands used to generate solutions in each part of this exercise. Save your M-file on your H drive and include a printout of your M-file with your examination. Place your login name and the filename of your M-file on the printout. I will be running your M-files from my office and grading them on performance. *Note*: Recall that Matlab's `figure` command opens a new figure window for plotting. Also, placing `close all` at the beginning of your file will close all open figure windows.

Exercise 2. Consider the following set of data.

x	y
2	582
3	1611
4	3206
5	5618
6	8760
7	12940
8	18159
9	24336
10	31615

(a) Use Matlab to plot the data. Use the appropriate plots to determine whether it is better to fit the data to an exponential model

$$y = ae^{bx},$$

or a power function having form

$$y = ax^b.$$

Justify your choice of model.

(b) Use a linear algebra least squares technique to fit your choice of model to the data. Provide two plots. The first plot should show the linear relationship found in the data set. The second plot should show a plot of the original data set and the plot of the model found that fits the data. Again, show all intermediate work, as in Exercise 1, on college ruled paper.

(c) Prepare an M-file that performs all of the necessary work, including the generation of all plots. Your M-file should use the `xlabel`, `ylabel`, and `title` commands to provide appropriate axes labels and a title containing the equation of the current plot. Save the file on your H drive. Obtain a printout of your M-file and include it with your examination papers. Place your login name and the filename of your M-file on the printout. I will be running these M-files from my office to grade their performance. *Note*: Recall that Matlab's `figure` command opens a new figure window for plotting. Also, placing `close all` at the beginning of your file will close all open figure windows.

Solutions to Exercises

Exercise 1(a) First, substitute each data point in the equation $y = at^2 + bt + c$.

$$\begin{aligned} 159 &= a(0)^2 + b(0) + c \\ 1489 &= a(5)^2 + b(5) + c \\ 2099 &= a(10)^2 + b(10) + c \\ 1936 &= a(15)^2 + b(15) + c \\ 898 &= a(20)^2 + b(20) + c \end{aligned}$$

This system of equations has matrix form

$$\begin{pmatrix} 0^2 & 0 & 1 \\ 5^2 & 5 & 1 \\ 10^2 & 10 & 1 \\ 15^2 & 15 & 1 \\ 20^2 & 20 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 159 \\ 1489 \\ 2099 \\ 1936 \\ 898 \end{pmatrix}.$$

Note that this system has the form $A\mathbf{x} = \mathbf{y}$, where

$$A = \begin{pmatrix} 0^2 & 0 & 1 \\ 5^2 & 5 & 1 \\ 10^2 & 10 & 1 \\ 15^2 & 15 & 1 \\ 20^2 & 20 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 159 \\ 1489 \\ 2099 \\ 1936 \\ 898 \end{pmatrix}.$$

Enter matrix the t-data in Matlab.

```
>> t
t =
     0
     5
    10
    15
    20
```

Now enter matrix A.

```
>> A=[t.^2,t,ones(size(t))]
```

```
A =
     0     0     1
    25     5     1
   100    10     1
   225    15     1
   400    20     1
```

Enter the vector $\mathbf{y}$.

```
>> y=[159;1489;2099;1936;898]
y =
          159
         1489
         2099
         1936
          898
```

Set up the normal equations by multiplying both sides of $A\mathbf{x} = \mathbf{y}$ by A^T.

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{y}$$

In Matlab, set up the augmented matrix.

```
 M=[A'*A,A'*y]
M =
      221250         12500           750       1041925
       12500           750            50         75435
         750            50             5          6581
```

Reduce.

```
>> R=rref(M)
R =
    1.0000         0         0  -15.7400
         0    1.0000         0  353.3000
         0         0    1.0000  144.2000
```

Therefore,

```
>> xhat=R(:,4)
xhat =
  -15.7400
  353.3000
  144.2000
```

and $a = -15.74$, $b = 353.3$, and $c = 144.2$. The best fit polynomial is

$$y = -15.74t^2 + 353.3t + 144.2.$$

□

Exercise 1(b) Prepare the polynomial data.

```
>> tt=linspace(-1,21)
>> yy=-15.74*tt.^2+353.3*tt+144.2
```

Plot the data and the best fit polynomial.

```
plot(t,y,'o',tt,yy)
```

Label the axes and provide a title.

```
>> xlabel('t-axis')
>> ylabel('y-axis')
>> title('The best fit polynomial y=-15.74t^2+353.3t+144.2')
```

These commands produce the image in Figure 1. □

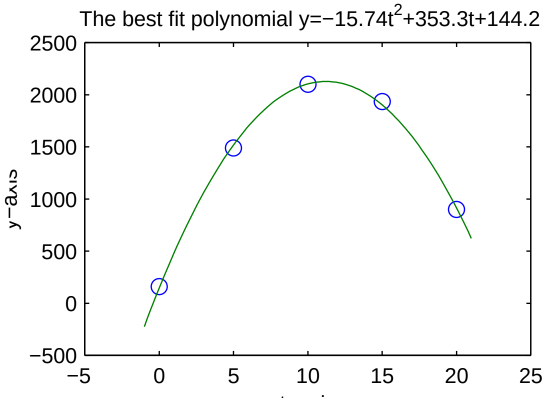

Figure 1: The best fit polynomial.

Exercise 1(c) Here is the M-file I used to prepare my solution. Note that I have carefully commented my code, always a good practice for programmers. At the end of the file, note that I have sized the graphic, then saved it in encapsulated postscript, color, level 2 postscript, for inclusion in my LaTeX document.

```
%close all existing figure windows
close all

%Enter data
t=(0:5:20).'
y=[159;1489;2099;1936;898]

%build matrix A
A=[t.^2,t,ones(size(t))]

%Craft  normal equations
M=[A'*A,A'*y]
```

```
%reduce, then strip xhat from last column
R=rref(M)
xhat=R(:,4)

%build best fit polynomial
tt=linspace(-1,21)
yy=-15.74*tt.^2+353.3*tt+144.2

%plot data and polynomial
plot(t,y,'o',tt,yy)

%labels and title
xlabel('t-axis')
ylabel('y-axis')
title('The best fit polynomial y=-15.74t^2+353.3t+144.2')

%change the paper size (4 inches by 3 inches)
```

```
%of the image prior to saving
set(gcf,'PaperPosition',[0,0,4,3])

%saving the image for inclusion in TeX document
print -depsc2 matlabexam1.eps
```

□

Exercise 2(a) First, enter the data in Matlab.

```
>> x=(2:10).'
>> y=[582;1611;3206;5618;8760;12940;18159;24336;31615]
```

If

$$y = ae^{bx},$$

then

$$\ln y = \ln ae^{bx},$$
$$\ln y = \ln a + \ln e^{bx},$$
$$\ln y = \ln a + bx.$$

Thus, if this is the correct model, then a graph of $\ln y$ versus x should produce a straight line. The following commands were used to craft the image shown in Figure 2.

```
>> plot(x,log(y),'.')
>> xlabel('x')
>> ylabel('log(y)')
>> title('A plot of log(y) versus x')
```

Note that the relation is nonlinear in Figure 2. Therefore, $y = ax^{bx}$ is not the correct model to use. If

$$y = ax^b,$$

then

$$\ln y = \ln ax^b,$$
$$\ln y = \ln a + \ln x^b,$$
$$\ln y = \ln a + b \ln x.$$

Thus, if this is the correct model to apply in this situation, the graph of $\ln y$ versus $\ln x$ should be a straight line. The following commands were used to create the image in Figure 3.

```
>> plot(log(x),log(y),'.')
>> xlabel('log(x)')
>> ylabel('log(y)')
>> title('A plot of log(y) versus log(x)')
```

Note that the relationship is linear. Therefore, $y = ax^b$ is the correct model to apply in this situation.

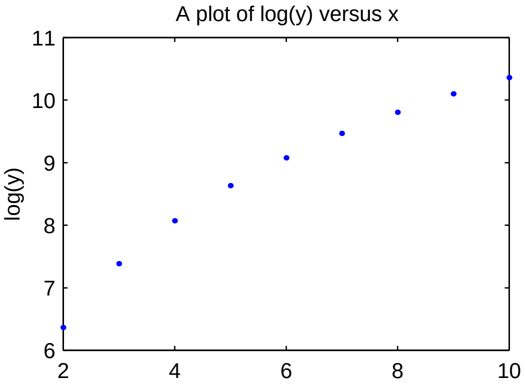

Figure 2: The plot of $\ln y$ versus x is not linear.

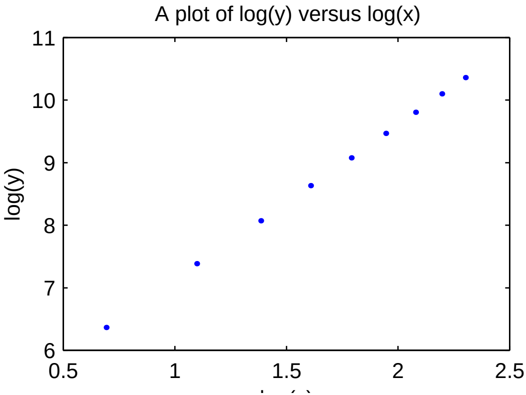

Figure 3: The plot of $\ln y$ versus $\ln x$ is linear.

□

Exercise 2(b) Because of our work in part (a), we know that we have to fit the data to the equation $\ln y = \ln a + b \ln x$. First, substitute the data points into this equation.

$$
\begin{aligned}
\ln 582 &= \ln a + b \ln 2 \\
\ln 1611 &= \ln a + b \ln 3 \\
\ln 3206 &= \ln a + b \ln 4 \\
\ln 5618 &= \ln a + b \ln 5 \\
\ln 8760 &= \ln a + b \ln 6 \\
\ln 12940 &= \ln a + b \ln 7 \\
\ln 18159 &= \ln a + b \ln 8 \\
\ln 24336 &= \ln a + b \ln 9 \\
\ln 31615 &= \ln a + b \ln 10
\end{aligned}
$$

This system of equations has matrix form

$$\begin{pmatrix} 1 & \ln 2 \\ 1 & \ln 3 \\ 1 & \ln 4 \\ 1 & \ln 5 \\ 1 & \ln 6 \\ 1 & \ln 7 \\ 1 & \ln 8 \\ 1 & \ln 9 \\ 1 & \ln 10 \end{pmatrix} \begin{pmatrix} \ln a \\ b \end{pmatrix} = \begin{pmatrix} \ln 582 \\ \ln 1611 \\ \ln 3206 \\ \ln 5618 \\ \ln 8760 \\ \ln 12940 \\ \ln 18159 \\ \ln 24336 \\ \ln 31615 \end{pmatrix}.$$

Note that this system has the form $A\mathbf{u} = \mathbf{v}$, where

$$A = \begin{pmatrix} 1 & \ln 2 \\ 1 & \ln 3 \\ 1 & \ln 4 \\ 1 & \ln 5 \\ 1 & \ln 6 \\ 1 & \ln 7 \\ 1 & \ln 8 \\ 1 & \ln 9 \\ 1 & \ln 10 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \ln a \\ b \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \ln 582 \\ \ln 1611 \\ \ln 3206 \\ \ln 5618 \\ \ln 8760 \\ \ln 12940 \\ \ln 18159 \\ \ln 24336 \\ \ln 31615 \end{pmatrix}.$$

Enter the matrix A in Matlab.

```
>> A=[ones(size(x)),log(x)]
```

Enter the vector $\mathbf{v}$.

```
>> v=log(y)
```

Set up the normal equations by multiplying both sides of $A\mathbf{u} = \mathbf{v}$ by A^T.

$$A^T A\hat{\mathbf{u}} = A^T \mathbf{v}$$

In Matlab, set up the augmented matrix.

```
>> M=[A'*A,A'*v]
M =
    9.0000   15.1044   79.2716
   15.1044   27.6502  138.7442
```

Reduce.

```
>> R=rref(M)
R =
    1.0000         0    4.6468
         0    1.0000    2.4794
```

Therefore,

```
>> uhat=R(:,3)
```

```
uhat =
    4.6468
    2.4794
```

and $\ln a = 4.6468$ and $b = 2.4794$. Thus,

$$\ln y = 4.6468 + 2.4794 \ln x.$$

Solve this equation for y.

$$y = e^{4.6468 + 2.4794 \ln x}$$
$$y = e^{4.6468} e^{2.4794 \ln x}$$
$$y = 104.2508 e^{\ln x^{2.4794}}$$
$$y = 104.2508 x^{2.4794}$$

We can now fit this model to the original data. First, build data satisfying the model.

```
>> xx=linspace(1,11);
>> yy=104.2508*xx.^(2.4794);
```

The following commands were used to craft the plot in Figure 4.

```
>> plot(x,y,'.',xx,yy)
>> xlabel('x-axis')
>> ylabel('y-axis')
>> title('The plot of y=104.2508x^{2.4794}')
```

□

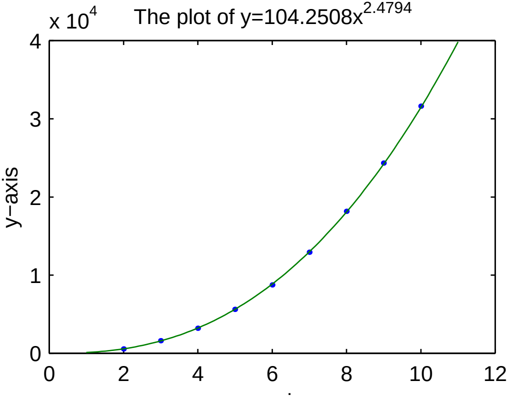

Figure 4: Fitting the data with a power function.

Exercise 2(c) Here is the M-file I used to produce the calculations and images for this exercise. Again, note the liberal use of comments and the sizing and saving of graphics for inclusion in a LaTeX document.

```
%close all existing figure windows
close all

%Enter data
x=(2:10).'
y=[582;1611;3206;5618;8760;12940;18159;24336;31615]

%plot log(y) versus t
plot(x,log(y),'.')

%label and title
xlabel('x')
ylabel('log(y)')
```

```
title('A plot of log(y) versus x')

%size and save the graphic
set(gcf,'PaperPosition',[0,0,3,2])
print -depsc2 matlabexam2a.eps

%plot log(y) versus t
plot(log(x),log(y),'.')

%label and title
xlabel('log(x)')
ylabel('log(y)')
title('A plot of log(y) versus log(x)')

%size and save the graphic
set(gcf,'PaperPosition',[0,0,3,2])
print -depsc2 matlabexam2b.eps
```

```
%create the matrix A
A=[ones(size(x)),log(x)]

%create the vector v
v=log(y)

%set up the augmented matrix for the normal equations
M=[A'*A,A'*v]

%reduce
R=rref(M)

%uhat
uhat=R(:,3)

%build data for the model
xx=linspace(1,11);
yy=104.2508*xx.^(2.4794);
```

```
%plot the data and model
plot(x,y,'.',xx,yy)

%provide axis labels and a title
xlabel('x-axis')
ylabel('y-axis')
title('The plot of y=104.2508x^{2.4794}')

%size and save the graphic
set(gcf,'PaperPosition',[0,0,4,3])
print -depsc2 matlabexam2c.eps
```

□