College of the Redwoods
Mathematics Department
Math 45—Linear Algebra

Pretest–Exam #3
Linear Algebra

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Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Complete the following definition.

A mapping $\mathcal{L}$ from a vector space V into a vector space W is a linear tr

Exercise 2. Let V, W be vector spaces and let $\mathcal{L} : V \to W$ be a linear transformation.

(a) Prove that $\mathcal{L}(\mathbf{0}) = \mathbf{0}$.

(b) Define $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathcal{L}(\mathbf{x}) = \mathbf{x} + \mathbf{a},$$

where $\mathbf{a} = (1, 2)^T$. Use part (a) to explain why this "translation" is not a linear transformation.

Exercise 3. Prove or disprove: $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathcal{L}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

is linear.

Exercise 4. Let V and W be vector spaces and $\mathcal{L} : V \to W$ a linear transformation.

(a) Complete the following definition:

The kernel of $\mathcal{L}$ is ...

(b) Define $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathcal{L}$ sends x to its projection onto the line $x_1 = x_2$. What is the kernel of $\mathcal{L}$?

Exercise 5. Let V, W be vector spaces and $\mathcal{L} : V \to W$ a linear transformation.

(a) Complete the following definition.

$\mathcal{L}$ is one-to-one iff ...

(b) Prove $\mathcal{L}$ is one-to-one iff $\ker(\mathcal{L}) = \{\mathbf{0}\}$.

Exercise 6. Let V, W be vector spaces and $\mathcal{L} : V \to W$ a linear transformation.

(a) Complete the following definition.

The mapping $\mathcal{L} : V \to W$ is onto W iff ...

(b) Let $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathcal{L}(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

Prove or disprove: $\mathcal{L}$ maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Exercise 7. Let $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in the plane through an angle θ about the origin. Find the "standard matrix representation" of $\mathcal{L}$.

Exercise 8. Define $\mathcal{L} : P_3 \to P_3$ by $\mathcal{L}(p(x)) = xp'(x)$. Let $\mathcal{B} = \{1, x, x^2\}$ be a basis for P_3. Determine a matrix representation for $\mathcal{L}$.

Exercise 9. Define $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathcal{L}(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

(a) Find the "standard matrix" of transformation.

(b) Use the standard matrix to find $\mathcal{L}(1, -1, 2)$.

(c) Let

$$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

be a basis for $\mathbb{R}^3$ and let

$$\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$

be a basis for $\mathbb{R}^2$. Find the matrix of transformation in these bases.

(d) Write $(1, -1, 2)^T$ in $\mathcal{U}$-coordinates, then use the transformation matrix in part (c) to find the $\mathcal{V}$-coordinates of the transformation of $(1, -1, 2)^T$. Compare this result to the solution found in part (b).

Exercise 10. Let $\mathcal{L}$ be the linear operator mapping $\mathbb{R}^3$ to $\mathbb{R}^3$ by $\mathcal{L}(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Let

$$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. Use the usual technique to find the matrix of transformation for this basis.

(b) Find the transition matrix U corresponding to a change of basis from $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and use it to determine the matrix B representing $\mathcal{L}$ with respect to $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Compare your solution with part(a).

Exercise 11. One final question on similar matrices.

(a) Complete the following definition.

Two matrices A and B are similar iff ...

(b) If A and B are similar, prove that

$$\det(A - \lambda I) = \det(B - \lambda I).$$

Solutions to Exercises

Exercise 1. *A* mapping $\mathcal{L}$ from a vector space V into a vector space W is a linear transformation iff . . .

1. $\mathcal{L}(\mathbf{u}+\mathbf{v}) = \mathcal{L}(\mathbf{u}) + \mathcal{L}(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \epsilon U$, and

2. $\mathcal{L}(\alpha\mathbf{u}) = \alpha\mathcal{L}(\mathbf{u})$ for all $\alpha\epsilon\mathbb{R}, \mathbf{u}\epsilon U$.

Exercise 1

Exercise 2(a) Since $\mathcal{L}$ is linear,

$$\begin{aligned}\mathcal{L}(\mathbf{0}) &= \mathcal{L}(\mathbf{0}+\mathbf{0}),\\ \mathcal{L}(\mathbf{0}) &= \mathcal{L}(\mathbf{0})+\mathcal{L}(\mathbf{0}).\end{aligned}$$

Subtract $\mathcal{L}(\mathbf{0})$ from both sides of this last equation and simplify.

$$\begin{aligned}\mathcal{L}(\mathbf{0})-\mathcal{L}(\mathbf{0}) &= \mathcal{L}(\mathbf{0})+\mathcal{L}(\mathbf{0})-\mathcal{L}(\mathbf{0})\\ \mathbf{0} &= \mathcal{L}(\mathbf{0})\end{aligned}$$

□

Exercise 2(b) If a transformation is linear, then $\mathcal{L}(\mathbf{0}) = \mathbf{0}$. However, in this case,

$$\mathcal{L}(\mathbf{0}) = \mathbf{0} + \mathbf{a} = \mathbf{a},$$

but $\mathbf{a} \neq \mathbf{0}$. Therefore, $\mathcal{L}(\mathbf{0}) \neq \mathbf{0}$ and the mapping cannot be linear.

□

Exercise 3. First,

$$\begin{aligned}
\mathcal{L}(\mathbf{u}+\mathbf{v}) &= \mathcal{L}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right), \\
&= \mathcal{L}\left(\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix}\right), \\
&= \begin{pmatrix} u_1+v_1 \\ (u_1+v_1)+(u_2+v_2) \\ (u_1+v_1)-(u_2+v_2) \end{pmatrix}, \\
&= \begin{pmatrix} u_1 \\ u_1+u_2 \\ u_1-u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_1+v_2 \\ v_1-v_2 \end{pmatrix}, \\
&= \mathcal{L}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + \mathcal{L}\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right), \\
&= \mathcal{L}(\mathbf{u}) + \mathcal{L}(\mathbf{v}).
\end{aligned}$$

Secondly,

$$\begin{aligned}
\mathcal{L}(\alpha\mathbf{u}) &= \mathcal{L}\left(\alpha\begin{pmatrix}u_1\\u_2\end{pmatrix}\right),\\
&= \mathcal{L}\left(\begin{pmatrix}\alpha u_1\\ \alpha u_2\end{pmatrix}\right),\\
&= \begin{pmatrix}\alpha u_1\\ \alpha u_1+\alpha u_2\\ \alpha u_1-\alpha u_2\end{pmatrix},\\
&= \alpha\begin{pmatrix}u_1\\ u_1+u_2\\ u_1-u_2\end{pmatrix},\\
&= \alpha\mathcal{L}\left(\begin{pmatrix}u_1\\u_2\end{pmatrix}\right),\\
&= \alpha\mathcal{L}(\mathbf{u}).
\end{aligned}$$

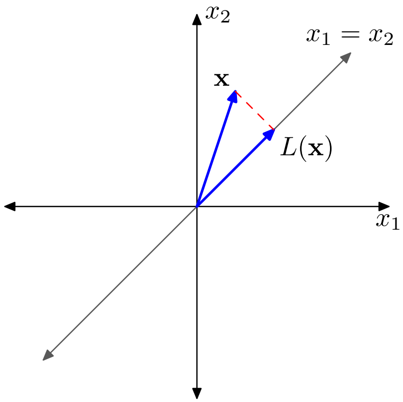
Exercise 3

Exercise 4(a) The kernel of $\mathcal{L}$ is defined as

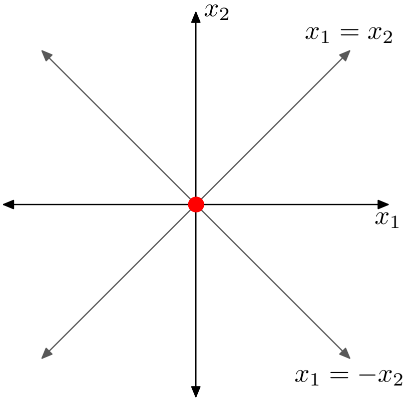
$$\ker(\mathcal{L}) = \{\mathbf{v} \epsilon V : \mathcal{L}(\mathbf{v}) = \mathbf{0}\}.$$

□

Exercise 4(b) Think geometrically. $\mathcal{L}$ sends any vector in the plane to its projection onto the line $x_1 = x_2$.



You must find all $\mathbf{x} \epsilon \mathbb{R}^2$ such that $\mathcal{L}(\mathbf{x}) = \mathbf{0}$. That is, you need to find all $\mathbf{x} \epsilon \mathbb{R}^2$ that are projected onto the zero vector. A little geometry reveals the answer.



Clearly, all points on the line $x_1 = -x_2$ are projected onto the origin. Thus, the kernel is the set of all points on the line $x_1 = -x_2$. □

Exercise 5(a) $\mathcal{L}$ is one-to-one iff $\mathcal{L}(\mathbf{v}_1) = \mathcal{L}(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$. Put another way, if $\mathbf{v}_1 \neq \mathbf{v}_2$, then $\mathcal{L}(\mathbf{v}_1) \neq \mathcal{L}(\mathbf{v}_2)$. □

Exercise 5(b) Proof:

($\Longrightarrow$) Assume $\mathcal{L}$ is one-to-one. If $\mathbf{x} \epsilon \ker(\mathcal{L})$, then

$$\mathcal{L}(\mathbf{x}) = \mathcal{L}(\mathbf{0}).$$

But, $\mathcal{L}(\mathbf{0}) = \mathbf{0}$, so we can write

$$\mathcal{L}(\mathbf{x}) = \mathcal{L}(\mathbf{0}).$$

But, $\mathcal{L}$ is one-to-one, so

$$\mathbf{x} = \mathbf{0}.$$

Therefore, the kernel of $\mathcal{L}$ consists solely of the zero vector; i.e., $\ker(\mathcal{L}) = \{\mathbf{0}\}$.

We must prove the converse.

($\Longleftarrow$) Assume $\ker(\mathcal{L}) = \{\mathbf{0}\}$. We need to show that $\mathcal{L}$ is one-to-one. Assume,

$$\mathcal{L}(\mathbf{v}_1) = \mathcal{L}(\mathbf{v}_2).$$

But, $\mathcal{L}$ is linear, so

$$\mathcal{L}(\mathbf{v}_1) - \mathcal{L}(\mathbf{v}_2) = \mathbf{0},$$
$$\mathcal{L}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}.$$

This puts $\mathbf{v}_1 - \mathbf{v}_2$ in the kernel of $\mathcal{L}$. But, the kernel consists solely of the zero vector, so

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0},$$

$$\mathbf{v}_1 = \mathbf{v}_2.$$

Therefore, $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathcal{L}$ is one-to-one. □

Exercise 6(a) The mapping $\mathcal{L} : V \to W$ is onto W iff for each $\mathbf{y} \epsilon W$, there exists an $\mathbf{x} \epsilon V$ such that $\mathcal{L}(\mathbf{x}) = \mathbf{y}$. □

Exercise 6(b) Let $\mathbf{y}$ be an arbitrary element in $\mathbb{R}^2$. We need to find an $\mathbf{x}\epsilon\mathbb{R}^2$ such that

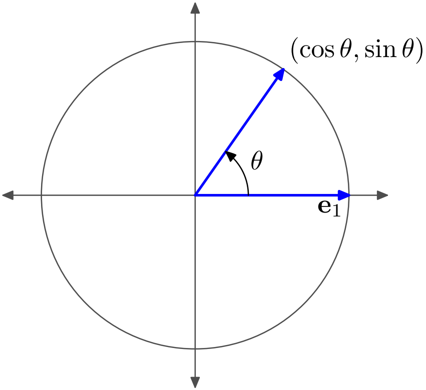
$$\mathcal{L}(\mathbf{x}) = \mathbf{y},$$
$$A\mathbf{x} = \mathbf{y},$$
$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Set up the augmented matrix and reduce.

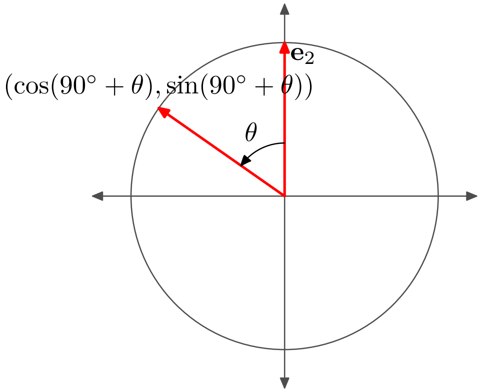
$$\begin{pmatrix} 2 & 4 & y_1 \\ 1 & 2 & y_2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & y_2 \\ 2 & 4 & y_1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & y_2 \\ 0 & 0 & y_1 - 2y_2 \end{pmatrix}$$

Unless $y_1 - 2y_2 = 0$, this system is inconsistent. Thus, $\mathcal{L}$ is not onto. Any point not on the line $y_1 - 2y_2 = 0$ will not have an $(x_1, x_2)\epsilon\mathbb{R}^2$ that maps onto it. □

Exercise 7. The transformation is completely determined by its action on the standard basis elements $\mathbf{e}_1$ and $\mathbf{e}_2$.



It's fairly clear that $\mathbf{e}_1$ is sent to the vector $(\cos\theta, \sin\theta)^T$. It's a little trickier to find the image of $\mathbf{e}_2$. But the following image is helpful.



However,

$$\cos(90^\circ + \theta) = \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta = -\sin\theta$$
$$\sin(90^\circ + \theta) = \sin 90^\circ \cos\theta + \sin\theta \cos 90^\circ = \cos\theta$$

Therefore, $\mathbf{e}_2$ gets sent to $(-\sin\theta, \cos\theta)^T$. Therefore,

$$A = [\mathcal{L}(\mathbf{e}_1)\ \mathcal{L}(\mathbf{e}_2)] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Exercise 7

Exercise 8. First, find the transformation of each basis element.

$$\mathcal{L}(1) = x \cdot 0 = 0$$
$$\mathcal{L}(x) = x \cdot 1 = x$$
$$\mathcal{L}(x^2) = x \cdot (2x) = 2x^2$$

Next, find the $\mathcal{B}$-coordinates of each transformation of each basis element.

$$\mathcal{L}(1) = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \Rightarrow [\mathcal{L}(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{L}(x) = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \Rightarrow [\mathcal{L}(x)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathcal{L}(x^2) = 0 \cdot 0 + 0 \cdot x + 0 \cdot 2x^2 \Rightarrow [\mathcal{L}(x)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Therefore, the matrix of transformation is

$$A = \begin{bmatrix} [\mathcal{L}(1)]_{\mathcal{B}} & [\mathcal{L}(x)]_{\mathcal{B}} & [\mathcal{L}(x^2)]_{\mathcal{B}} \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Exercise 8

Exercise 9(a) Determine the action of $\mathcal{L}$ on each of the standard basis elements for $\mathbb{R}^3$.

$$\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(1,0,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{e}_2) = \mathcal{L}(0,1,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{e}_3) = \mathcal{L}(0,0,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Therefore, the standard matrix of transformation is

$$A = [\mathcal{L}(\mathbf{e}_1) \;\; \mathcal{L}(\mathbf{e}_2) \;\; \mathcal{L}(\mathbf{e}_3)] = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$\square$

Exercise 9(b)

$$\mathcal{L}(1,-1,2) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

□

Exercise 9(c) First determine the action of $\mathcal{L}$ on each basis element in $\mathcal{U}$.

$$\mathcal{L}(\mathbf{u}_1) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{u}_2) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{u}_1) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Determine the $\mathcal{V}$-coordinates of each of these results. To find the $\mathcal{V}$-coordinates of $(0,3), (1,2)$, and $(1,1)$, set up the augmented matrix.

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 2 & 1 \end{pmatrix}$$

And reduce.

$$\begin{pmatrix} 1 & 0 & 3/2 & 1 & 1/2 \\ 0 & 1 & 3/2 & 0 & -1/2 \end{pmatrix}$$

Thus,

$$B = \Big[[\mathcal{L}(\mathbf{u}_1)]_{\mathcal{V}} \;\; [\mathcal{L}(\mathbf{u}_2)]_{\mathcal{V}} \;\; [\mathcal{L}(\mathbf{u}_3)]_{\mathcal{V}} \Big]$$

$$B = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 3/2 & 0 & -1/2 \end{pmatrix}.$$

□

Exercise 9(d) To find the $\mathcal{U}$-coordinates of $(1, -1, 2)^T$, set up the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

and reduce.

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Therefore,

$$\left[\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right]_{\mathcal{U}} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Now,

$$\mathcal{B} \left[\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right]_{\mathcal{U}} = \left[\mathcal{L} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right]_{\mathcal{V}}.$$

Therefore,

$$\left[\mathcal{L}\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]_{\mathcal{V}} = \begin{pmatrix}3/2 & 1 & 1/2\\3/2 & 0 & -1/2\end{pmatrix}\begin{pmatrix}2\\-3\\2\end{pmatrix},$$
$$= \begin{pmatrix}1\\2\end{pmatrix}.$$

Thus,

$$\mathcal{L}\begin{pmatrix}1\\-1\\2\end{pmatrix} = 1\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix},$$

precisely the answer found in part (b). □

Exercise 10(a) Take $\mathcal{L}$ of each basis element in $\mathcal{U}$.

$$\mathcal{L}(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathcal{L}(\mathbf{u}_3) = A\mathbf{u}_3 = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

To find the $\mathcal{U}$-coordinates of $\mathcal{L}(\mathbf{u}_1), \mathcal{L}(\mathbf{u}_2),$ and $\mathcal{L}(\mathbf{u}_3)$, set up the augmented matrix

$$[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathcal{L}(\mathbf{u}_1)\ \mathcal{L}(\mathbf{u}_2)\ \mathcal{L}(\mathbf{u}_3)], \begin{pmatrix} 1 & 1 & -1 & -1 & 3 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ -1 & 0 & 1 & -1 & 2 & 2 \end{pmatrix},$$

and reduce.

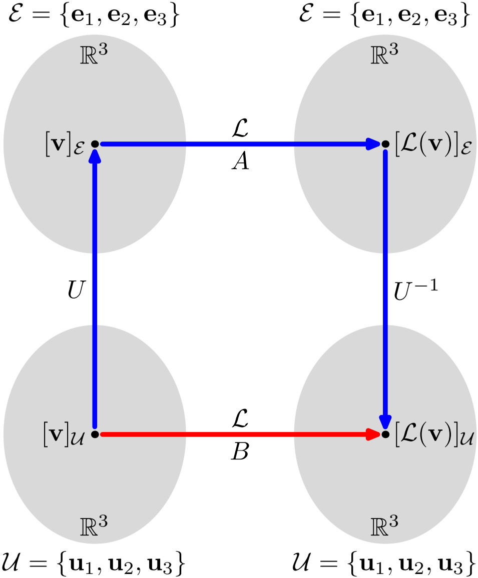
$$\begin{pmatrix} 1 & 0 & 0 & 2 & -7 & -4 \\ 0 & 1 & 0 & -2 & 5 & 4 \\ 0 & 0 & 1 & 1 & -5 & -2 \end{pmatrix}$$

Therefore, the matrix of transformation is

$$B = \Big[[\mathcal{L}(\mathbf{u}_1)]_{\mathcal{U}} \;\; [\mathcal{L}(\mathbf{u}_2)]_{\mathcal{U}} \;\; [\mathcal{L}(\mathbf{u}_3)]_{\mathcal{U}} \Big] = \begin{pmatrix} 2 & -7 & -4 \\ -2 & 5 & 4 \\ 1 & -5 & -2 \end{pmatrix}.$$

$\square$

Exercise 10(b) Draw this picture.


$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
$\mathbb{R}^3$
$\mathbb{R}^3$
$[\mathbf{v}]_\mathcal{E}$
$\mathcal{L}$
A
$[\mathcal{L}(\mathbf{v})]_\mathcal{E}$
U
U^{-1}
$[\mathbf{v}]_\mathcal{U}$
$\mathcal{L}$
B
$[\mathcal{L}(\mathbf{v})]_\mathcal{U}$
$\mathbb{R}^3$
$\mathbb{R}^3$
$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$
$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

Now,

$$U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

According to the commutative diagram,

$$B = U^{-1}AU,$$

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & -7 & -4 \\ -2 & 5 & 4 \\ 1 & -5 & -2 \end{pmatrix},$$

which is the same matrix of transformation found in part (a).

□

Exercise 11(a) Two matrices A and B are similar iff there exists an invertible matrix U such that $A = U^{-1}BU$. □

Exercise 11(b) Proof:

If A and B are similar, then there exists an invertible matrix U such that

$$A = U^{-1}BU.$$

Then,

$$\begin{aligned} A - \lambda I &= U^{-1}BU - \lambda I \\ &= U^{-1}BU - I(\lambda I) \\ &= U^{-1}BU - U^{-1}U(\lambda I). \end{aligned}$$

But the identity matrix I commutes with U and the scalar λ can be placed anywhere. Therefore

$$\begin{aligned} A - \lambda I &= U^{-1}BU - U^{-1}(\lambda I)U, \\ &= U^{-1}[B - \lambda I]U. \end{aligned}$$

Therefore,

$$\det(A - \lambda I) = \det(U^{-1}[B - \lambda I]U).$$

But the determinant of a product is the product of the determinants and since U is invertible, $\det(U^{-1}) = 1/\det(U)$.

$$\begin{aligned}\det(A - \lambda I) &= \det(U^{-1})\det(B - \lambda I)\det(U) \\ &= \frac{1}{\det(U)}\det(B - \lambda I)\det(U) \\ &= \det(B - \lambda I)\end{aligned}$$

□