College of the Redwoods
Mathematics Department
Math 45—Linear Algebra

Exam #3
Linear Algebra

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Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit. Feel free to use a calculator or Matlab on this examination. If you are more comfortable with hand calculations, that is fine, as the exam is not that calculation intensive.*

EXERCISE 1. Let V and W be vector spaces over $\mathbb{R}$ and T a function that maps V to W. That is, $T : V \mapsto W$.

(a) Complete the following definition.

> T is a *linear transformation* if and only if ...

(b) As an example, define $R^{2\times 2}$ as the set of all 2×2 matrices with real entries. Define $T : \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ by $T(A) = \operatorname{tr}(A)$, where $\operatorname{tr}(A)$ denotes the *trace* of the matrix A. Prove or disprove: T is a linear transformation.

Exercise 2. Find the "standard matrix" of transformation for each of the following linear transformations.

(a) $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by T sends (x, y) to its reflection across the line $y = -x$.

(b) $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by T sends (x, y) to its projection onto the x-axis.

(c) $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by T rotates (x, y) $\pi/4$ radians in a counterclockwise direction.

Exercise 3. Let V and W be vector spaces and $T : V \mapsto W$ be a linear transformation.

(a) Complete the following definition.

> T is a *onto* W if and only if ...

(b) Which of the transformations in Exercise 2 are onto $\mathbb{R}^2$?

Exercise 4. Let V and W be vector spaces and let $T : V \mapsto W$ be a linear transformation.

(a) Complete the following definition.

T is *one-to-one* if and only if ...

(b) Define $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}.$$

Prove or disprove: T is one-to-one.

Exercise 5. Let V and W be vector spaces and let $T : V \mapsto W$ be a linear transformation.

(a) Complete the following definition.

> The *kernel* of T is . . .

(b) What is the kernel of the transformation in Exercise 4(b)?

Exercise 6. Define $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by

$$T(x, y) = \begin{pmatrix} -x + y \\ x + 2y \end{pmatrix}.$$

(a) Find the "standard matrix" of the transformation T. Let A represent this matrix.

(b) Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

be a basis for $\mathbb{R}^2$. Find the matrix of the transformation T in terms of the basis $\mathcal{B}$. Let C represent this matrix.

(c) Let B represent the transition matrix, the matrix that will map $\mathcal{B}$-coordinates to the standard basis coordinates. Express matrix C in terms of the matrices A and B and check your result. Show the work for your check.

Solutions to Exercises

Exercise 1(a) T is a linear transformation iff

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \epsilon V$.
2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\alpha \epsilon \mathbb{R},\ \mathbf{u} \epsilon V$.

□

Exercise 1(b) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ by $T(A) = \text{tr}(A)$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be arbitrary elements in $\mathbb{R}^{2\times 2}$.

1.

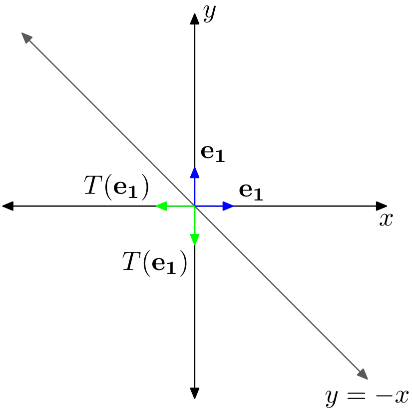
$$
\begin{aligned}
T(A+B) &= \operatorname{tr}(A+B) \\
&= \operatorname{tr}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}\right) \\
&= (a_{11}+b_{11}) + (a_{22}+b_{22}) \\
&= (a_{11}+a_{22}) + (b_{11}+b_{22}) \\
&= \operatorname{tr}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) + \operatorname{tr}\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \\
&= \operatorname{tr}(A) + \operatorname{tr}(B) \\
&= T(A) + T(B)
\end{aligned}
$$

2. Let $\alpha \epsilon \mathbb{R}$

$$\begin{aligned}
T(\alpha A) &= \mathrm{tr}(\alpha A) \\
&= \mathrm{tr}\left(\alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \\
&= \mathrm{tr}\left(\begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}\right) \\
&= \alpha a_{11} + \alpha a_{22} \\
&= \alpha(a_{11} + a_{22}) \\
&= \alpha \mathrm{tr}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \\
&= \mathrm{tr}(A) \\
&= \alpha T(A)
\end{aligned}$$

Therefore, $T(A) = \mathrm{tr}(A)$ is a linear transformation. □

Exercise 2(a) T reflects each point across the line $y = -x$.



Thus,

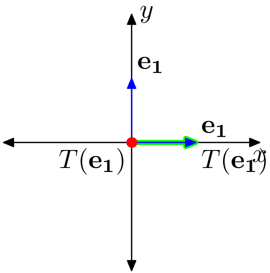
$$T(\mathbf{e}_1) = -\mathbf{e}_2,$$
$$T(\mathbf{e}_2) = -\mathbf{e}_1.$$

Hence, the standard matrix of transformation is

$$\begin{aligned} A &= [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)], \\ &= [-\mathbf{e}_2 \;\; -\mathbf{e}_1], \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

□

Exercise 2(b) T projects each point onto the x-axis.



Thus,

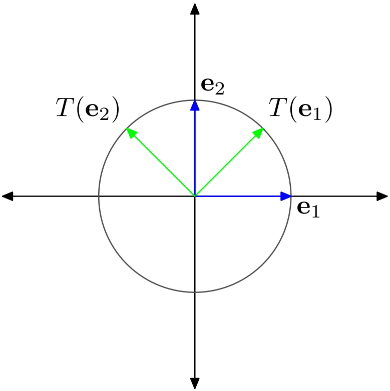
$$T(\mathbf{e}_1) = \mathbf{e}_1,$$
$$T(\mathbf{e}_2) = 0.$$

Hence the standard matrix of transformation is

$$\begin{aligned} A &= [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)], \\ &= [\mathbf{e}_1 \ 0], \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

□

Exercise 2(c) T rotates each point $\pi/4$ radians in counterclockwise direction.



Thus,

$$T(\mathbf{e}_1) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix},$$
$$T(\mathbf{e}_2) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Hence the standard matrix of transformation is

$$\begin{aligned} A &= [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)], \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}. \end{aligned}$$

□

Exercise 3(a) T is onto W iff for each $\mathbf{w} \epsilon V$ there exists a $\mathbf{v} \epsilon V$ such that $T(\mathbf{v}) = \mathbf{w}$. □

Exercise 3(b) The reflection map is onto. For each $(x, y)^T \epsilon \mathbb{R}^2$, there is a point on the other side of the line $y = -s$ that maps onto $(x, y)^T$.

The projection map is not onto R^2. For example, start with $(1, 1)^T \epsilon \mathbb{R}^2$. Obviously, there is no point in R^2 that maps onto $(1, 1)^T$, simply because all points are mapped onto the x-axis.

The rotation map is onto. Take any point $(x, y)^T$ in $\mathbb{R}^2$ and rotate the point 45° in the clockwise direction. This will be the point that maps onto $(x, y)^T$. □

Exercise 4(a) T is one-to-one if and only if $\mathbf{v}_1 \neq \mathbf{v}_2$ implies that $T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$. □

Exercise 4(b) Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Clearly, $\mathbf{v}_1 \neq \mathbf{v}_2$, but

$$T(\mathbf{v}_1) = T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \text{ and}$$

$$T(\mathbf{v}_2) = T\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

Thus, $\mathbf{v}_1 \neq \mathbf{v}_2$, but $T(\mathbf{v}_1) = T(\mathbf{v}_2)$. Therefore, T is not one-to-one.

□

Exercise 5(a) The kernel of T is the set of all $\mathbf{v}\epsilon V$ such that $T(\mathbf{v}) = 0$. In symbols,

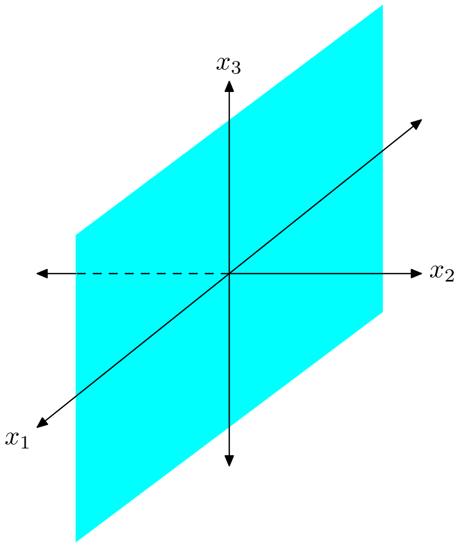
$$\ker(T) = \{\mathbf{v}\epsilon V : T(\mathbf{v}) = \mathbf{0}\}.$$

□

Exercise 5(b) The mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$$

projects all points onto the $x_1 x_3$-plane by “forgetting” that x_2-coordinate. With this geometry in mine, it is easy to see that all points on the x_2-axis map to the zero vector.


x_3
x_2
x_1

Alternatively, we can find the kernel by solving the equation

$$T(x_1, x_2, x_3) = \mathbf{0},$$
$$\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $x_1 = x_3 = 0$, but x_2 is free to be any value. This is precisely the definition of the x_2-axis. □

Exercise 6(a) First, find the image of each "standard basis" vector.

$$T(\mathbf{e}_1) = T(1,0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_2) = T(0,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Therefore, the "standard matrix" of the transformation is

$$\begin{aligned} A &= [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

□

Exercise 6(b) First, find the image of each basis element in

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

$$T(\mathbf{b}_1) = A\mathbf{b}_1 = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

$$T(\mathbf{b}_2) = A\mathbf{b}_2 = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

Next, find the $\mathcal{B}$-coordinates of each image of each basis element. This is easily accomplished by reducing.

$$\begin{pmatrix} 1 & 1 & -3 & -2 \\ -2 & -1 & -3 & -1 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & 6 & 3 \\ 0 & 1 & -9 & -5 \end{pmatrix}$$

Thus,

$${}_{\mathcal{B}} = \begin{pmatrix} 6 \\ -9 \end{pmatrix},$$

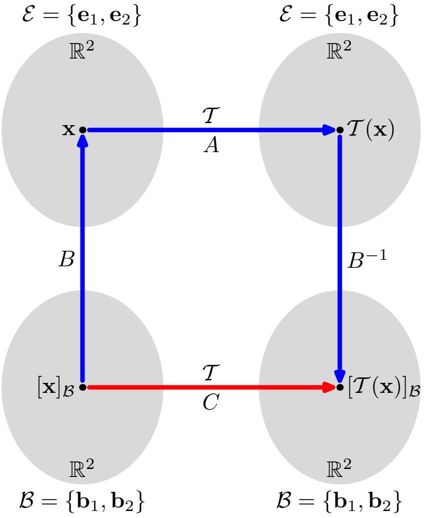
$$[T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -5 \end{pmatrix},$$

and the matrix of transformation becomes

$$C = \left[[T(\mathbf{b}_1)]_{\mathcal{B}} \;\; [T(\mathbf{b}_2)]_{\mathcal{B}} \right] = \begin{pmatrix} 6 & 3 \\ -9 & -5 \end{pmatrix}.$$

□

Exercise 6(c)



$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$
$\mathbb{R}^2$
$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$
$\mathbb{R}^2$
$\mathbf{x}$
$\mathcal{T}$
A
$\mathcal{T}(\mathbf{x})$
B
B^{-1}
$[\mathbf{x}]_{\mathcal{B}}$
$\mathcal{T}$
C
$[\mathcal{T}(\mathbf{x})]_{\mathcal{B}}$
$\mathbb{R}^2$
$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$
$\mathbb{R}^2$
$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$

The transition matrix is

$$B = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

It is clear from the commutative diagram that

$$C = B^{-1}AB.$$

Checking,

$$\begin{aligned} B^{-1}AB &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \\ &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ -3 & -1 \end{pmatrix}, \\ &= \begin{pmatrix} 6 & 3 \\ -9 & -5 \end{pmatrix}, \\ &= C. \end{aligned}$$

□