College of the Redwoods
Mathematics Department
Math 45—Linear Algebra

Exam #1—Chapter One

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Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Consider the system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4, \\ x_2 + x_3 &= 6, \\ 2x_1 + 3x_3 &= 5. \end{aligned}$$

(a) Set up the augmented matrix representing the system.

(b) Using hand calculations only, place the augmented matrix in row-echelon form (**not** reduced row echelon form).

(c) Write the system of equations represented by the augmented matrix (row echelon form) from part (b).

(d) Use back substitution to solve the system from part (c).

EXERCISE 2. The system

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 - x_3 &= 4 \\ 3x_1 + x_2 &= 10 \end{aligned}$$

has an augmented matrix having reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -1/2 & 2 \\ 0 & 1 & 3/2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Write the solution to the system.

Exercise 3. Matrix A is

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Matrix B has dimensions 3×24 (3 rows and 24 columns). Every entry in matrix B is a 1. That is, if $B = (b_{ij})$, the $b_{ij} = 1$ for all i and j. What is the first column of the matrix product AB?

Exercise 4. You are given matrices A, B, and C:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Use the appropriate technique to find the inverse of matrix A. That is, find A^{-1}.

(b) Solve the equation

$$AX + B = C,$$

where X is a 3×3 matrix of unknown entries.

Exercise 5. Let A and B be $n \times n$ nonsingular matrices. Prove that AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Exercise 6. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}.$$

(a) Write A as a product of a lower triangular matrix L (with ones on the main diagonal) and an upper triangular matrix U.

(b) Check your result from part (a). That is, using hand calculations only, verify that $A = LU$.

Solutions to Exercises

Exercise 1(a)

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 6 \\ 2 & 0 & 3 & 5 \end{pmatrix}$$

□

Exercise 1(b) Multiply row 1 by -2 and add to row 3.

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 6 \\ 0 & -2 & -1 & -3 \end{pmatrix}$$

Multiply row 2 by 2 and add to row 3.

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

□

Exercise 1(c)

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \quad (1) \\ x_2 + x_3 &= 6 \quad (2) \\ x_3 &= 9 \quad (3) \end{aligned}$$

□

Exercise 1(d) Sub $x_3 = 9$ in (2)

$$\begin{aligned} x_2 + 9 &= 6 \\ x_2 &= -3 \end{aligned}$$

Sub $x_3 = 9$ and $x_2 = -3$ in (1).

$$\begin{aligned} x_1 + (-3) + 2(9) &= 4 \\ x_1 - 3 + 18 &= 4 \\ x_1 + 15 &= 4 \\ x_1 &= -11 \end{aligned}$$

□

Exercise 2. First, write the equations represented by the reduced row echelon form of the augmented matrix.

$$x_1 - \frac{1}{2}x_3 = 2$$
$$x_2 + \frac{3}{2}x_3 = 4$$

Identify the pivot variables and free variables: x_1 and x_2 are pivot variables, x_3 is a free variable. Solve each equation for its pivot variable.

$$x_1 = 2 + \frac{1}{2}x_3$$
$$x_2 = 4 - \frac{3}{2}x_3$$

Let the free variable be represented by a variable of choice and substitute their choice in the remaining equations. Thus, the solution is

$$x_1 = 2 + \frac{1}{2}\alpha,$$

$$x_2 = 4 - \frac{3}{2}\alpha,$$

$$x_3 = \alpha,$$

where α is any real number.

Exercise 2

Exercise 3. Let $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_{24}$ represent the columns of matrix B. Then

$$\begin{aligned} AB &= A[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_{24}] \\ &= [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \ldots \ A\mathbf{b}_{24}] \end{aligned}$$

Therefore, the first column of the product AB is $A\mathbf{b}_1$, where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus,

$$\begin{aligned} A\mathbf{b}_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\ &= 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \\ &= \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}. \end{aligned}$$

Exercise 3

Exercise 4(a) Form the augmented matrix

$$[A\ I] = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Use elementary row operations to transform $[A\ I]$ into the form $[I\ A^{-1}]$. Multiply row 1 by -2 and add to row 3.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 & 1 \end{pmatrix}$$

Multiply row 2 by -1.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & 1 \end{pmatrix}$$

Multiply row 3 by $-1/2$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{pmatrix}$$

Multiply row 3 by 1 and add to row 2. Multiply row 3 by -1 and add to row 1.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{pmatrix}$$

Multiply row 2 by -1 and add to row 1.

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{pmatrix}$$

Thus,

$$A^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1/2 \\ 1 & 0 & -1/2 \end{pmatrix}$$

□

Exercise 4(b) Subtract B from both sides of the equation.

$$AX = C - B$$

Multiply on the left by A^{-1}.

$$\begin{aligned} A^{-1}(AX) &= A^{-1}(C - B) \\ (A^{-1}A)X &= A^{-1}(C - B) \\ IX &= A^{-1}(C - B) \\ X &= A^{-1}(C - B) \end{aligned}$$

Substitute A^{-1}, C, and B and perform the matrix operations.

$$X = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1/2 \\ 1 & 0 & -1/2 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right)$$

$$X = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1/2 \\ 1 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 1 & -1 \\ 3/2 & -3/2 & 3/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}$$

□

Exercise 5. We need to show that $B^{-1}A^{-1}$ is both a left inverse and a right inverse of AB. Note that $B^{-1}A^{-1}$ exists because A and B are nonsingular (invertible).

First, show that $B^{-1}A^{-1}$ is a left inverse of AB.

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I\end{aligned}$$

Next, show that $B^{-1}A^{-1}$ is a right inverse of AB.

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Therefore, AB is nonsingular and its inverse is given by

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Exercise 5

Exercise 6(a)

Use only the third elementary row operation to place A in upper triangular form, recording multipliers in L along the way.

Thus, multiply row 1 by -2 and add to row 3.

$$E_1A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Multiply row 2 by $-1/2$ and add to row 3.

$$E_2E_1A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1/2 \end{pmatrix} = U \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1/2 & 1 \end{pmatrix}$$

□

Exercise 6(b)

$$\begin{aligned} LU &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix} \\ &= A \end{aligned}$$

□