The Dimension of a Vector Space

Math 45 — Linear Algebra
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Abstract. If the set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent and span the vector space V, then the set B is called a *basis* for the vector space V. The number of vectors in the set B is called the *dimension* of the vector space V. In this activity you will use a number of *Matlab* commands to explore and reinforce these ideas.
Prerequisites. Familiarity with Matlab's `rref` command and its use to solve linear systems is assumed. Readers should also be familiar with the concepts of linear independence and the span of a set of vectors.

Introduction

If the set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent and span the vector space V, then the set B is called a *basis* for the vector space V. The number of vectors in the set B is called the *dimension* of the vector space V. In this activity you will use a number of *Matlab* commands to explore and reinforce these ideas.

The Null Space

The *null space* of a matrix A is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. In symbols,

$$Nul(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

It's often easiest to describe the null space of a matrix by finding a basis for the null space. Before trying an example, first set *Matlab's* format for rational arithmetic.

```
>> format rat
```

Example *Find a basis for the null space of the matrix*

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 2 & 0 & 3 & -2 & 4 \end{bmatrix}$$

Solution. Load the matrix A into the *Matlab* workspace.

```
>> A=[1 -2 1 0 1;-1 1 0 -2 2;2 0 3 -2 4]
A =
     1    -2     1     0     1
    -1     1     0    -2     2
     2     0     3    -2     4
```

The null space is composed of all solutions of the matrix equation $A\mathbf{x} = \mathbf{0}$. I strongly suggest that you set up the problem as shown in equation (1).

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 2 & 0 & 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

Note that the solutions of equation (1) are elements of R^5. The null space of matrix A is a subspace of R^5. Set up an augmented matrix for the system in equation (1).

```
>> M=[A, [0 0 0]']
M =
     1    -2     1     0     1     0
    -1     1     0    -2     2     0
     2     0     3    -2     4     0
```

Place this matrix in reduced row echelon form with *Matlab*'*s* `rref` command.

```
>> rref(M)
ans =
      1            0            0            2        -11/5            0
      0            1            0            0         -1/5            0
      0            0            1           -2         14/5            0
```

The reduced row echelon form of the matrix `M` indicates that the solutions of equation (1) can be written

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_4 + \frac{11}{5}x_5 \\ \frac{1}{5}x_5 \\ 2x_4 - \frac{14}{5}x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_4 \\ 0 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{11}{5}x_5 \\ \frac{1}{5}x_5 \\ -\frac{14}{5}x_5 \\ 0 \\ x_5 \end{bmatrix}$$

$$= x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{11}{5} \\ \frac{1}{5} \\ -\frac{14}{5} \\ 0 \\ 1 \end{bmatrix} \qquad (2)$$

where x_4 and x_5 are any real numbers. Equation (2) guarantees that all solutions of the equation $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination of the vectors $\mathbf{v}_1 = (-2, 0, 2, 1, 0)$ and

$\mathbf{v}_2 = (11/5, 1/5, -14/5, 0, 1)$. Since the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent and they span the solution set of $A\mathbf{x} = \mathbf{0}$, they form a basis for the solutions of $A\mathbf{x} = \mathbf{0}$. Therefore,

$$\mathrm{B}_1 = \left\{ \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{11}{5} \\ \frac{1}{5} \\ -\frac{14}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the null space of the matrix A. Because this basis for the null space of A has two vectors, the *dimension* of the null space is 2. The dimension of the null space of the matrix A is called the *nullity* of A.

Bases Are Not Unique

The null space of the matrix A can have several different bases. Now that you know that the dimension of the null space is 2, any two independent vectors from the null space of matrix A can serve as a basis. For example, in equation (2), if you let $x_4 = 1$ and $x_5 = 0$, then let $x_4 = 0$ and $x_5 = 5$, then

$$\mathrm{B}_2 = \left\{ \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 1 \\ -14 \\ 0 \\ 5 \end{bmatrix} \right\}$$

is also a basis for the null space of the matrix A (the vectors are linearly independent and span the set of solutions of $A\mathbf{x} = \mathbf{0}$).

Matlab's Null Command

Matlab's `null` command, when used on numeric objects, find a very special basis for the null space of the matrix A.

```
>> help null

 NULL   Null space.
    Z = NULL(A) is an orthonormal basis for the null space of A obtained
    from the singular value decomposition.  That is,  A*Z has negligible
    elements, size(Z,2) is the nullity of A, and Z'*Z = I.

    Z = NULL(A,'r') is a "rational" basis for the null space obtained
    from the reduced row echelon form.  A*Z is zero, size(Z,2) is an
    estimate for the nullity of A, and, if A is a small matrix with
    integer elements, the elements of R are ratios of small integers.

    The orthonormal basis is preferable numerically, while the rational
    basis may be preferable pedagogically.
```

The following command produces an *orthonormal* basis for the null space.

```
>> format
>> N=null(A)

N =
```

```
   -0.6668         0
    0.0026    0.1238
    0.6591   -0.3714
    0.3475    0.6809
    0.0128    0.6190
```

The columns of the matrix N are the basis vectors. These vectors have unit length and are orthogonal (perpendicular) to each other. Hence the word *orthonormal*.

You can produce the basis B_1 with the following Matlab commands.

```
>> format rat
>> N=null(A,'r')

N =

     -2          11/5
      0           1/5
      2         -14/5
      1             0
      0             1
```

The Column Space

The column space of a matrix A is simply defined as the span of the columns of A. That is, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of the matrix A, then the column space of the matrix A is defined as follows:

$$col(A) = Span\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$$

Because the set $C = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ already spans the column space, simply delete dependent vectors one at a time until you have a linearly independent set. For example, consider again the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 2 & 0 & 3 & -2 & 4 \end{bmatrix}$$

The column space of A is the span of its columns

$$col(A) = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$$

The reduced row echelon form of A is

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -2 & \frac{14}{5} \end{bmatrix}$$

If you label the column vectors of the matrix U with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4,$ and $\mathbf{u}_5$, then it is easy to see that $\mathbf{u}_4 = 2\mathbf{u}_1 - 2\mathbf{u}_3$. It is important to note that the columns of A have the same dependency relation; that is, $\mathbf{a}_4 = 2\mathbf{a}_1 - 2\mathbf{a}_3$. You can eliminate $\mathbf{a}_4$ and the remaining vectors will still span the column space of matrix A.

$$col(A) = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}$$

Secondly, it is easy to see that $\mathbf{u}_5 = -\frac{11}{5}\mathbf{u}_1 - \frac{1}{5}\mathbf{u}_2 + \frac{14}{5}\mathbf{u}_3$. The columns of A will have the same dependency relation, $\mathbf{a}_5 = -\frac{11}{5}\mathbf{a}_1 - \frac{1}{5}\mathbf{a}_2 + \frac{14}{5}\mathbf{a}_3$. You can eliminate $\mathbf{a}_5$ and the remaining vectors will still span the column space of the matrix *A*.

$$col(A) = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$$

Because $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ will also be independent and the set

$$B_3 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

is a basis for the column space of *A*. Because this basis has 3 vectors, the dimension of the column space of the matrix *A* is three. The dimension of the column space of the matrix *A* is also called the *rank* of the matrix *A*.

The *Matlab* command that best imitates this process is the `rref` command. You will need to read the help file again, only this time pay closer attention than when you first read this file.

```
>> help rref

 RREF   Reduced row echelon form.
    R = RREF(A) produces the reduced row echelon form of A.

    [R,jb] = RREF(A) also returns a vector, jb, so that:
        r = length(jb) is this algorithm's idea of the rank of A,
        x(jb) are the bound variables in a linear system, Ax = b,
        A(:,jb) is a basis for the range of A.
```

The range of *A* and the column space of *A* are different terms with the same meaning. Consequently, the following commands will capture a basis for the column space of *A*.

```
>> [R,jb]=rref(A)

R =

     1            0            0            2        -11/5
     0            1            0            0         -1/5
     0            0            1           -2         14/5


jb =

     1            2            3
```

Because `jb=[1 2 3]`, columns 1, 2, and 3 form a basis for the column space of the matrix *A*. You can capture this basis with a little fancy use of *Matlab*'*s* indexing power.

```
>> B3=A(:,jb)
B3 =
     1           -2            1
    -1            1            0
     2            0            3
```

Note that this result is identical to the basis B_3 you found earlier.

Once again, bases are not unique. Any three independent vectors that span the column space of matrix *A* can serve as a basis for *A*.

Matlab's Orth Command

Matlab's `orth` command produces a very special basis for the column space of matrix *A*.

```
>> help orth

 ORTH   Orthogonalization.
```

```
    Q = ORTH(A) is an orthonormal basis for the range of A.
>> format
>> C=orth(A)

C =

    0.2497   -0.6217   -0.7424
    0.3096    0.7777   -0.5471
    0.9175   -0.0932    0.3867
```

The columns of matrix *C* form a basis for the column space (or range) of matrix *A*. The columns of matrix *C* are unit vectors (length one) that are mutually orthogonal. Hence the term *orthonormal* basis.

Homework Exercises

1. Prepare a table with headings as follows:

Matrix A	Number of Columns	Basis for Nul(A)	Nullity	Basis for Col(A)	Rank

Use *Matlab* to complete column entries in your table for each of the following matrices:

1. a. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ -1 & 2 & 2 & -2 & 4 \\ 0 & 1 & 4 & -2 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ -1 & 1 & 3 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 2 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 1 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 2 & -2 & 4 & 0 \end{bmatrix}$

Based on your tabular results, what can you say about the relationship between Number of Columns, Nullity, and Rank?