College of the Redwoods
Mathematics Department
Math 45 — Linear Algebra

Exam #2
Vector Spaces

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Essay Questions

Read Carefully! *You have the weekend to complete the exam. The exam is due, on my desk, at the beginning of class on Monday.*

This exam is open notes, open book. You may use a calculator or computer to check your work where appropriate. You must answer all of the exercises on your own. You are not allowed to work in groups on the exam. You are not allowed to enlist the aid of a tutor or friend to help with the exam. You are not allowed to read the exercises in the exam, then seek help on similar questions. Once you open the exam and read the questions, you may not seek any outside help of any kind. From the moment you open the exam, you must do everything by yourself.

This includes online examinations that I have given in the past. Up to the point that you open your examination and read the questions, you are free to peruse past examinations on my website. But, once you've opened your examination, you may no longer refer to any former examinations and/or solutions. However, open book, open notes still stand, even after you have opened the exam and read the *questions.*

Place the solution to each exercise on a separate sheet of paper. On a good sheet of paper, write out (longhand) and sign the following honor pledge.

> I promise that all work found herein is my own. I have received no help from tutors, colleagues, or other teachers. I have honored all of the examination constraints listed in the directions.

Arrange the problems in order, place these examination pages on top of your solutions, then place the honor pledge on top of the examination as a cover sheet. Staple. Good luck!

EXERCISE 1. Let A be an arbitrary $n \times n$ matrix.

(a) Define what is meant by a *symmetric* matrix, then prove that the matrix $B = A + A^T$ is symmetric.

(b) Define what is mean by a *skew-symmetric* matrix, then prove that $C = A - A^T$ is skew-symmetric.

Exercise 2. Consider the following subset of R^2.

$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x + 2y = 0 \right\}$$

Prove that H is a subspace of R^2 by showing that

(a) H is closed under addition, and

(b) H is closed under scalar multiplication.

Exercise 3. Consider the matrix

$$A = \begin{pmatrix} 0 & 3 & 0 & 3 & -3 \\ -3 & -2 & 3 & 1 & 2 \\ -2 & 1 & 2 & 3 & -1 \end{pmatrix}$$

and its reduced row echelon form

$$R = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for the column space of A. What is the dimension of the column space?

(b) Find a basis for the nullspace of A. What is the dimension of the null space?

EXERCISE 4. Construct a matrix A whose nullspace consists only of multiples of the vector $(1, -2, 1)^T$.

EXERCISE 5. Construct a matrix whose column space contains the vector $(1, 2, -1)^T$ and whose nullspace contains the vectors $(1, 0, 1)^T$ and $(2, 1, 0)^T$.

EXERCISE 6. A system $A\mathbf{x} = \mathbf{b}$ having augmented matrix $[A\ \mathbf{b}]$ is row equivalent to the reduced row echelon form

$$[R\ \mathbf{d}] = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) By inspection, find a "particular" solution $\mathbf{x}_p$ of system $A\mathbf{x} = \mathbf{b}$.

(b) By inspection, find the nullspace or "special solutions" of matrix A.

(c) Write the "complete" solution of the system $A\mathbf{x} = \mathbf{b}$.

Exercise 7. Find q so that matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & q \\ 1 & 1 & 1 \end{pmatrix}$$

has rank 2. *Note: An answer with no supportive work and prose will receive no credit.*

Exercise 8. Find three vectors that are linearly dependent, such that any two of the three vectors are independent. Prove that your solution satisfies the requirement of the exercise. That is,

(a) show that the three vectors are dependent, and

(b) show that any two of the three are independent.

Exercise 9. Let M be the set of all 2×3 matrices with real entries. You may assume that M is a vector space over the real numbers (it is). Find a basis $\mathcal{B}$ for the vector space M. *Note: Bases are not unique. There are several possible choices here.*

(a) Prove that the elements in your set $\mathcal{B}$ are linearly independent.

(b) Prove that the elements in your set $\mathcal{B}$ span M, the set of all 2×3 matrices with real entries.

(c) What is the dimension of the vector space M? Why?

Solutions to Exercises

Exercise 1(a) A matrix A is *symmetric* if and only if $A^T = A$. Let $B = A + A^T$. Then,

$$\begin{aligned} B^T &= (A + A^T)^T \\ &= A^T + (A^T)^T \\ &= A^T + A \\ &= A + A^T \\ &= B. \end{aligned}$$

Because $B^T = B$, B is symmetric.

□

Exercise 1(b) A matrix A is *skew-symmetric* if and only if $A^T = -A$. Let $C = A - A^T$. Then,

$$\begin{aligned} C^T &= (A - A^T)^T \\ &= A^T - (A^T)^T \\ &= A^T - A \\ &= -(A - A^T) \\ &= -C. \end{aligned}$$

Because $C^T = -C$, C is skew-symmetric.

□

Exercise 2(a) Set

$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \; x + 2y = 0 \right\}.$$

We need to show that H is closed under addition. Pick $\mathbf{u}$ from H. Then,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{and} \qquad u_1 + 2u_2 = 0.$$

Pick $\mathbf{v}$ from H. Then,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{and} \qquad v_1 + 2v_2 = 0.$$

We need to show that $\mathbf{u} + \mathbf{v}$ is back in H. But,

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

and

$$\begin{aligned}(u_1+v_1)+2(u_2+v_2) &= u_1+v_1+2u_2+2v_2\\ &= (u_1+2u_2)+(v_1+2v_2)\\ &= 0+0\\ &= 0.\end{aligned}$$

Thus, $\mathbf{u}+\mathbf{v}$ is back in H and H is closed under addition.

□

Exercise 2(b) We need to show that H is closed under scalar multiplication. Let $\mathbf{u} \in H$ and $\alpha \in R$. Then,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{and} \qquad u_1 + 2u_2 = 0.$$

Then,

$$\alpha\mathbf{u} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix},$$

and

$$\begin{aligned} \alpha u_1 + 2(\alpha u_2) &= \alpha(u_1 + 2u_2) \\ &= \alpha(0) \\ &= 0. \end{aligned}$$

Therefore, $\alpha\mathbf{u} \in H$ and H is closed under scalar multiplication.

□

Exercise 3(a) Matrix

$$A = \begin{pmatrix} 0 & 3 & 0 & 3 & -3 \\ -3 & -2 & 3 & 1 & 2 \\ -2 & 1 & 2 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R,$$

so columns 1 and 2 are pivot columns. Thus,

$$\mathcal{B}_{C(A)} = \left\{ \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\}$$

is a basis for the column space of A. Because the basis has 2 independent vectors, the dimension of the column space (the rank) is 2.

$\square$

Exercise 3(b) Consider again

$$R = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, and $\mathbf{r}_5$ denote the columns of R, which has free columns $\mathbf{r}_3$, $\mathbf{r}_4$,and $\mathbf{r}_5$. Note that

$$1\mathbf{r}_1 + 0\mathbf{r}_2 + 1\mathbf{r}_3 + 0\mathbf{r}_4 + 0\mathbf{r}_5 = \mathbf{0},$$

making

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

a "special" solution of $R\mathbf{x} = \mathbf{0}$. Secondly,

$$1\mathbf{r}_1 - 1\mathbf{r}_2 + 0\mathbf{r}_3 + 1\mathbf{r}_4 + 0\mathbf{r}_5 = \mathbf{0},$$

so

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is a second "special" solution. Finally,

$$0\mathbf{r}_1 + 1\mathbf{r}_2 + 0\mathbf{r}_3 + 0\mathbf{r}_4 + 1\mathbf{r}_5 = \mathbf{0},$$

so

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a final "special" solution. Thus,

$$\mathcal{B}_{N(A)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $N(A)$. The dimension of the nullspace is 3.

□

Exercise 4. Because a basis for the nullspace of matrix A is

$$\mathcal{B}_{N(A)} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\},$$

our choice of matrix must have only one free variable, but 3 columns. There are many such matrices, but we will choose

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Note $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and note that

$$1\mathbf{a}_1 - 2\mathbf{a}_2 + 1\mathbf{a}_3 = \mathbf{0}.$$

Thus, $(1, -2, 1)^T$ is in the nullspace of our matrix A.

Exercise 4

Exercise 5. We start by insuring that $(1, 2, -1)^T$ is in the column space by making this vector one of the columns of A.

$$A = \begin{pmatrix} 1 & x & x \\ 2 & x & x \\ -1 & x & x \end{pmatrix}$$

Note that A must be 3×3 as $(1, 0, 1)^T$ and $(2, 1, 0)^T$ must be in the nullspace of A. Now, to have $(2, 1, 0)^T$ in $N(A)$, we need

$$2\mathbf{a}_1 + 1\mathbf{a}_2 = \mathbf{0},$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are columns 1 and 2 of matrix A. Thus,

$$\mathbf{a}_2 = -2\mathbf{a}_1 = -2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

Matrix A is now

$$A = \begin{pmatrix} 1 & -2 & x \\ 2 & -4 & x \\ -1 & 2 & x \end{pmatrix}.$$

But $(1, 0, 1)^T$ is in the nullspace, so

$$1\mathbf{a}_1 + 1\mathbf{a}_3 = \mathbf{0},$$

whre $\mathbf{a}_1$ and $\mathbf{a}_3$ are columns 1 and 3 of matrix A. Thus,

$$\mathbf{a}_3 = -\mathbf{a}_1 = -1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Thus,

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ -1 & 2 & 1 \end{pmatrix}.$$

Exercise 5

Exercise 6(a) Examine

$$[R\ \mathbf{d}] = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and note that columns 2 and 4 are free columns. Letting the free variables $x_2 = x_4 = 0$, we get $x_1 = 3$, $x_3 = -2$, and $x_5 = 1$. Thus,

$$\mathbf{x}_p = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

is a "particular" solution of $A\mathbf{x} = \mathbf{b}$.

□

Exercise 6(b) Let $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, and $\mathbf{r}_5$ denote the columns of

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that columns 2 and 4 are "free" columns. Further, note that

$$-2\mathbf{r}_1 + 1\mathbf{r}_2 + 0\mathbf{r}_3 + 0\mathbf{r}_4 + 0\mathbf{r}_5 = \mathbf{0},$$

so

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a basis element for the nullspace of R, and hence also for A. Secondly,

$$-2\mathbf{r}_1 + 0\mathbf{r}_2 - 2\mathbf{r}_3 + 1\mathbf{r}_4 + 0\mathbf{r}_5 = \mathbf{0},$$

so

$$\begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

is a second basis element for $N(A)$. Hence,

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for the nullspace of A.

□

Exercise 6(c) The complete solution is the sum of one particular solution and an arbitrary element from the nullspace.

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$$

But $\mathbf{x}_n$, being in $N(A)$, can be written as a linear combination of the basis elements for $N(A)$. Hence,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

is the "complete" solution, with α and β being any real nubmers.

□

Exercise 7. Reduce. Subtract 1 times row 1 from row 3.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & q \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & q \\ 0 & -1 & 2 \end{pmatrix}$$

Exchange rows 2 and 3.

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 3 & q \end{pmatrix}$$

Multiply row 2 by -1.

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 3 & q \end{pmatrix}$$

Subtract 3 times row 2 from row 3.

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & q+6 \end{pmatrix} = U$$

To have rank 2, matrix U must have exactly one free variable. This will be the case only if

$$\begin{aligned} q + 6 &= 0 \\ q &= -6. \end{aligned}$$

Exercise 7

Exercise 8(a) There are many solutions. Let's try

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Note that

$$V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$$

has 1 free variable so the vectors are dependent. Indeed, $(-1, -1, 1)^T$ is in the nullspace of V, so

$$-1\mathbf{v}_1 - 1\mathbf{v}_2 + 1\mathbf{v}_3 = \begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This nontrivial combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ equals the zero vector. Hence, the vectors are dependent.

□

Exercise 8(b) However, each pair of vectors is independent. For example,

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{is not a multiple of} \quad \begin{pmatrix}0\\1\\0\end{pmatrix},$$

so these columns are independent. Secondly,

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{is not a multiple of} \quad \begin{pmatrix}1\\1\\0\end{pmatrix},$$

so these columns are independent. Finally,

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{is not a multiple of} \quad \begin{pmatrix}1\\1\\0\end{pmatrix},$$

so these columns are also independent.

□

Exercise 9(a) Let M be the set of all 2×3 matrices with real entries. Let

$$\mathcal{B} = \left\{ \begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \right\}$$

I claim that the elements of $\mathcal{B}$ are linearly independent. Consider

$$\alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ \alpha_4 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, adding,

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ and the "vectors" in $\mathcal{B}$ are linearly independent.

□

Exercise 9(b) Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

be an arbitrary element in M, the set of all 2×3 matrices with real entries. Then,

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, any element in M can be written as a linear combination of the elements in $\mathcal{B}$. Hence, $\mathcal{B}$ spans M and is a basis for M.

□

Exercise 9(c) Because the basis $\mathcal{B}$ for M has 6 elements, M has dimension 6.

□