College of the Redwoods
Mathematics Department
Math 45 — Linear Algebra

Exam #3
Linear Algebra

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Essay Questions

Read Carefully! *You have the weekend to complete the exam. The exam is due, on my desk, at the beginning of class on Wednesday.*

This exam is open notes, open book. You may use a calculator or computer to check your work where appropriate. You must answer all of the exercises on your own. You are not allowed to work in groups on the exam. You are not allowed to enlist the aid of a tutor or friend to help with the exam. You are not allowed to read the exercises in the exam, then seek help on similar questions. Once you open the exam and read the questions, you may not seek any outside help of any kind. From the moment you open the exam, you must do everything by yourself.

Place the solution to each exercise on a separate sheet of paper. On a good sheet of paper, write out (longhand) and sign the following honor pledge.

> I promise that all work found herein is my own. I have received no help from tutors, colleagues, or other teachers. I have honored all of the examination constraints listed in

the directions.

Arrange the problems in order, place these examination pages on top of your solutions, then place the honor pledge on top of the examination as a cover sheet. Staple. Good luck!

EXERCISE 1. Let S be the space of solutions to the equation

$$x_1 - x_2 + x_3 - 2x_4 = 0.$$

(a) Find a basis for the space S.

(b) Use the Gram-Schmidt orthogonalization process and the vectors found in part (a) to find an *orthonormal* basis for the space S.

(c) Use the basis found in part (b) to find the vector in S that is closest to the vector $\mathbf{b} = (1, 1, 1, 1)^T$.

Exercise 2. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 7 & 5 \end{pmatrix}.$$

(a) Use Gaussian elimination to find the determinant of A.

(b) Use the permutation definition to find the determinant of A.

(c) Use the cofactor expansion to find the determinant of A.

Exercise 3. Let A and B be 3×3 matrices with $|A| = 4$ and $|B| = -3$.

(a) Evaluate $|A^T A|$.

(b) Evaluate $|AB^{-1}|$.

(c) Evaluate $|-3B|$.

(d) If P is an invertible 3×3 matrix, evaluate $\det(PAP^{-1})$.

Exercise 4. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix A, then λ must either be 0 or 1.

Exercise 5. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}.$$

All work is to be done by hand, pencil and paper calculations only.

Exercise 6. Consider the matrix

$$A = \begin{pmatrix} 5 & -3 & 4 \\ 0 & 2 & 0 \\ -8 & 8 & -7 \end{pmatrix}.$$

(a) Using pencil and paper calculations only, find the characteristic polynomial for matrix A.

(b) Use a graphing calculator to draw the characteristic polynomial and find its roots. These are the eigenvalues. Make a copy of the graph and its roots on your examination paper.

(c) Using pencil and paper calculations only, compute the eigenvectors of matrix A.

Exercise 7. Consider the matrix

$$A = \begin{pmatrix} .7 & .2 \\ .3 & .8 \end{pmatrix}.$$

(a) Diagonalize the matrix A. That is, find an invertible S and a diagonal matrix D so that $A = SDS^{-1}$.

(b) Use the result from part (a) to find A^k. Write a 2×2 matrix with correct entries for this result.

(c) Use the result from part (c) to find

$$\lim_{k \to \infty} A^k.$$

Solutions to Exercises

Exercise 1(a) Let S be the space of solutions to the equation

$$x_1 - x_2 + x_3 - 2x_4 = 0.$$

In matrix from,

$$\begin{pmatrix} 1 & -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.$$

Thus, we find a basis for the null space of

$$A = \begin{pmatrix} 1 & -1 & 1 & -2 \end{pmatrix}.$$

Note that column 2 is a multiple of column 1.

$$\text{col } 2 = -1 \cdot \text{col } 1$$

Thus,

$$1 \cdot \text{col } 1 + 1 \cdot \text{col } 2 + 0 \cdot \text{col } 3 + 0 \cdot \text{col } 4 = 0$$

and

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

is a basis element for $N(A)$. Similarly,

$$\text{col } 3 = 1 \cdot \text{col } 1.$$

Thus,

$$-1 \cdot \text{col } 1 + 0 \cdot \text{col } 2 + 1 \cdot \text{col } 3 + 0 \cdot \text{col } 4 = 0$$

and

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is a second basis element for $N(A)$. Finally,

$$\text{col } 4 = -2 \cdot \text{col } 1$$

Thus,

$$2 \cdot \text{col } 1 + 0 \cdot \text{col } 2 + 0 \cdot \text{col } 3 + 1 \cdot \text{col } 4 = 0$$

and

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a final basis element for $N(A)$. Thus,

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the null space of A or the solution space of $x_1 - x_2 + x_3 - 2x_4 = 0$. □

Exercise 1(b) Let

$$a = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let

$$A = a = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let

$$\begin{aligned} B &= b - \frac{b \cdot A}{A \cdot A} A \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

To ease computation, let

$$B = 2B = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

Note that this new value of B is still orthogonal to A. Next,

$$\begin{aligned} C &= c - \frac{c \cdot A}{A \cdot A} A - \frac{c \cdot B}{B \cdot B} B \\ &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{-2}{6} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \\ 1 \end{pmatrix} \end{aligned}$$

Again, to ease computation, let

$$C = 3C = \begin{pmatrix} 2 \\ -2 \\ 2 \\ 1 \end{pmatrix}.$$

Check that C is orthogonal to both A and B.

$$A \cdot C = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 2 \\ 3 \end{pmatrix} = 0 \quad \text{and} \quad B \cdot C = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 2 \\ 3 \end{pmatrix} = 0.$$

Divide each vector by its length to get an orthonormal basis.

$$q_A = \frac{A}{\|A\|} = \frac{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}{\sqrt{2}} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}$$

$$q_B = \frac{B}{\|B\|} = \frac{\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}}{\sqrt{6}} = \begin{pmatrix}-1/6\\1/\sqrt{6}\\2/\sqrt{6}\\0\end{pmatrix}$$

$$q_C = \frac{C}{\|C\|} = \frac{\begin{pmatrix}2\\-2\\2\\3\end{pmatrix}}{\sqrt{21}} = \begin{pmatrix}2/\sqrt{21}\\-2/\sqrt{21}\\2/\sqrt{21}\\3/\sqrt{21}\end{pmatrix}$$

□

Exercise 1(c) To find the vector in S that is closest to $b = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$, project b onto S. To find the projection matrix,

$$P = Q(Q^T Q)^{-1} Q^T$$
$$P = QIQ^T$$
$$P = QQ^T$$

Set

$$Q = [q_A, q_B, q_C] = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{pmatrix}.$$

Then,

$$P = QQ^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 2/\sqrt{21} & -2/\sqrt{21} & 2/\sqrt{21} \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 6 & 1 & -1 & 2 \\ 1 & 6 & 1 & -2 \\ -1 & 1 & 6 & 2 \\ 2 & -2 & 2 & 3 \end{pmatrix}$$

Thus, the vector in S closest to $b = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}^T$ is

$$\begin{aligned}
P &= QQ^T b \\
&= \frac{1}{7}\begin{pmatrix} 6 & 1 & -1 & 2 \\ 1 & 6 & 1 & -2 \\ -1 & 1 & 6 & 2 \\ 2 & -2 & 2 & 3 \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \\
&= \frac{1}{7}\begin{pmatrix}8\\6\\8\\5\end{pmatrix} \\
&= \begin{pmatrix}8/7\\6/7\\8/7\\5/7\end{pmatrix}.
\end{aligned}$$

□

Exercise 2(a) If

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 7 & 5 \end{pmatrix},$$

subtract 3 times row 1 from row 2. Subtract 1 times row 1 from row 3.

$$\to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -5 \\ 0 & 8 & 3 \end{pmatrix}$$

Subtract 2 times row 2 from row 3

$$\to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 13 \end{pmatrix} = U$$

The determinant of U is the product of the entries on its diagonal. Thus,

$$|A| = |U| = (1)(4)(13) = 52$$

□

Exercise 2(b) Given

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 7 & 5 \end{pmatrix},$$

then

$$\begin{aligned} |A| &= \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 7 & 5 \end{pmatrix} \\ &= (1)(1)(5) - (1)(1)(7) + (-1)(1)(1) - (-1)(3)(5) + (2)(3)(7) - (2)(1) \\ &= 5 - 7 - 1 + 15 + 42 - 2 \\ &= 62 - 10 \\ &= 52 \end{aligned}$$

□

Exercise 2(c) Using cofactors and expanding across the first row,

$$\begin{aligned}
|A| &= \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 7 & 5 \end{pmatrix} \\
&= 1 \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix} \\
&= 1(5-7) + 1(15-1) + 2(21-1) \\
&= -2 + 14 + 40 \\
&= 52
\end{aligned}$$

□

Exercise 3(a) Matrix A is 3×3 with $|A| = 4$. The determinant of a product is the product of the determinants.

$$|A^T A| = |A^T||A|$$

But the determinant of the transpose is identical to the determinant of the matrix.

$$|A^T A| = |A^T||A| = |A||A| = 4 \cdot 4 = 16$$

□

Exercise 3(b) If $|B| = -3$, then we can write

$$1 = |I| = |BB^{-1}| = |B||B^{-1}|.$$

Thus,

$$|B^{-1}| = \frac{1}{|B|} = \frac{1}{-3} = -\frac{1}{3}.$$

Then

$$|AB^{-1}| = |A||B^{-1}| = 4\left(-\frac{1}{3}\right) = -\frac{4}{3}.$$

□

Exercise 3(c) Each time you multiply a row by -3, you multiply the determinant by -3. But B is 3×3, so $-3B$ results in *three* rows being multiplied by -3. Thus,

$$|-3B| = (-3)^3|B| = -27(-3) = 81.$$

□

Exercise 3(d) If P is invertible, then $|P| \neq 0$ and

$$\begin{aligned} |PAP^{-1}| &= |P||A||P^{-1}| \\ &= |P||A| \cdot \frac{1}{|P|} \\ &= |A| \\ &= 4. \end{aligned}$$

□

Exercise 4. We know that A in $n \times n$ and idempotent; i.e., $A^2 = A$. Let λ be an eigenvalue of A. Then there is a nonzero vector x such that

$$Ax = \lambda x.$$

Multiply both sides of this equation by A.

$$\begin{aligned} A(Ax) &= A(\lambda x) \\ (AA)x &= \lambda(Ax) \\ A^2x &= \lambda(Ax). \end{aligned}$$

But $A^2 = A$ and $Ax = \lambda x$, so

$$\begin{aligned} Ax &= \lambda(\lambda x) \\ \lambda x &= \lambda^2 x \\ 0 &= \lambda^2 x - \lambda x \\ 0 &= (\lambda^2 - \lambda)x \end{aligned}$$

But $x \neq 0$. Therefore,

$$\lambda^2 - \lambda = 0$$
$$\lambda(\lambda - 1) = 0$$

Thus, $\lambda = 0$ or $\lambda = 1$.

Exercise 4

Exercise 5. We need non-trivial solutions of

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

The nullspace of $A - \lambda I$ will be non-trivial only if

$$\begin{aligned} |A - \lambda I| &= 0 \\ \left| \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| &= 0 \\ \begin{vmatrix} 7-\lambda & -10 \\ 5 & -8-\lambda \end{vmatrix} &= 0 \\ (7-\lambda)(-8-\lambda) + 50 &= 0 \\ -56 - 7\lambda + 8\lambda + \lambda^2 + 50 &= 0 \\ \lambda^2 + \lambda - 6 &= 0 \\ (\lambda + 3)(\lambda - 2) &= 0 \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$. For $\lambda_1 = -3$,

$$\begin{aligned} A - \lambda_1 I &= A + 3I \\ &= \begin{pmatrix} 10 & -10 \\ 5 & -5 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Hence,

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector associated with $\lambda_1 = -3$. For $\lambda_2 = 2$,

$$\begin{aligned} A - \lambda_2 I &= A - 2I \\ &= \begin{pmatrix} 5 & -10 \\ 5 & -10 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}. \\ v_2 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

is an eigenvector associated with $\lambda_2 = 2$. We summarize the eigenvalue-eigenvector pairs

$$\begin{aligned} \lambda_1 = -3 \leftrightarrow v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = 2 \leftrightarrow v_2 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

Exercise 5

Exercise 6(a) If

$$A = \begin{pmatrix} 5 & -3 & 4 \\ 0 & 2 & 0 \\ -8 & 8 & -7 \end{pmatrix},$$

then

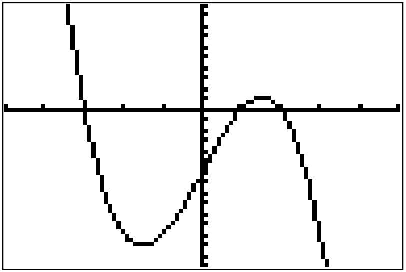
$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 & 4 \\ 0 & 2-\lambda & 0 \\ -8 & 8 & -7-\lambda \end{vmatrix}$$

Expanding by cofactors across the second row,

$$\begin{aligned} |A - \lambda I| &= (2-\lambda) \begin{vmatrix} 5-\lambda & 4 \\ -8 & -7-\lambda \end{vmatrix} \\ &= (2-\lambda)[(5-\lambda)(-7-\lambda) + 32] \\ &= (2-\lambda)(-35 - 5\lambda + 7\lambda + \lambda^2 + 32) \\ &= (2-\lambda)(\lambda^2 + 2\lambda - 3) \\ &= (2-\lambda)(\lambda+3)(\lambda-1) \end{aligned}$$

□

Exercise 6(b) The characteristic polynomial, $p(\lambda) = (2-\lambda)(\lambda+3)(\lambda-1)$, has this graph.



The zeros are 2, -3, and 1. These are the eigenvalues. □

Exercise 6(c) For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} 3 & -3 & 4 \\ 0 & 0 & 0 \\ -8 & 8 & -9 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 4/3 \\ -8 & 8 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 4/3 \\ 0 & 0 & 5/3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 4/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is an eigenvector associated with $\lambda_1 = 2$. For $\lambda_2 = -3$,

$$
\begin{aligned}
A + 3I &= \begin{pmatrix} 8 & -3 & 4 \\ 0 & 5 & 0 \\ -8 & 8 & -4 \end{pmatrix} \\
&\to \begin{pmatrix} 8 & -3 & 4 \\ 0 & 5 & 0 \\ 0 & 5 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 8 & -3 & 4 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & -3/8 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Thus,

$$v_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

is an eigenvector associated with $\lambda_2 = -3$. For $\lambda_3 = 1$,

$$
\begin{aligned}
A - I = \begin{pmatrix} 4 & -3 & 4 \\ 0 & 1 & 0 \\ -8 & 8 & -8 \end{pmatrix}
&\to \begin{pmatrix} 4 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 4 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Thus,

$$v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenvector associated with $\lambda_3 = 1$.

□

Exercise 7(a) First find the characteristic polynomial of

$$A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D,$$

where

$$\begin{aligned} T &= tr(A) = 1.5 \\ D &= det(A) = 0.56 - 0.06 = 0.5 \end{aligned}$$

Hence, the characteristic equation is

$$\begin{aligned} \lambda^2 - 1.5\lambda + 0.5 &= 0 \\ 10\lambda^2 - 15\lambda + 5 &= 0 \\ 2\lambda^2 - 3\lambda + 1 &= 0 \\ (2\lambda - 1)(\lambda - 1) &= 0. \end{aligned}$$

Thus, the eigenvalues are 1/2 and 1. With $\lambda_1 = 1/2$,

$$
\begin{aligned}
A - \frac{1}{2}I &= A - 0.5I \\
&= \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{pmatrix} \\
&\to \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus,

$$
v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
$$

is an eigenvector associated with $\lambda_1 = 1/2$. With $\lambda_2 = 1$,

$$A - I = \begin{pmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{pmatrix}$$
$$\to \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$$
$$\to \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

is an eigenvector associated with $\lambda_2 = 1$. Thus,

$$A = SDS^{-1}$$
$$= \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}^{-1}.$$

□

Exercise 7(b)

$$
\begin{aligned}
A^k &= (SDS^{-1})^k \\
&= SD^kS^{-1} \\
&= \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} (1/2)^k & 0 \\ 0 & 1^k \end{pmatrix} \left(-\frac{1}{5}\right) \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix} \\
&= -\frac{1}{5} \begin{pmatrix} -(1/2)^k & 2(1)^k \\ (1/2)^k & 3(1)^k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix} \\
&= -\frac{1}{5} \begin{pmatrix} -(1/2)^k & 2 \\ (1/2)^k & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix} \\
&= -\frac{1}{5} \begin{pmatrix} -2-3(1/2)^k & -2+2(1/2)^k \\ -3+3(1/2)^k & -3-2(1/2)^k \end{pmatrix}.
\end{aligned}
$$

□

Exercise 7(c) Thus,

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} \left(-\frac{1}{5}\right) \begin{pmatrix} -2-3(1/2)^k & -2+2(1/2)^k \\ -3+3(1/2)^k & -3-2(1/2)^k \end{pmatrix}$$
$$= \begin{pmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{pmatrix}$$

□