Dynamical Systems—Long Term Behavior

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Abstract

The closed form solution of difference equations is used to predict the long-term behavior of dynamical systems. The role of the "dominant" eigenvalue is introduced.

Prerequisites. Eigenvalues and eigenvectors. You should first complete the activity "Difference Equations—Closed Form Solutions" before attempting this activity.

Required M-files. Download a copy of `traj2.m`.

1 Long Term Behavior of the Sequence λ^n

We need to thoroughly investigate the behavior of the sequence of numbers generated by the expression λ^n. We will need to consider several cases.

1.1 When λ Is a Real Number

Let's perform several experiments where λ is a real number. For example, what is the limiting value of $(0.5)^n$ as $n \to +\infty$? The following Matlab code generates the first 15 terms of the sequence $(0.5)^n$.

```
>> n=(1:15)';

>> (0.5).^n

ans =

    0.5000
    0.2500
    0.1250
    0.0625
    0.0312
    0.0156
    0.0078
    0.0039
    0.0020
```

```
    0.0010
    0.0005
    0.0002
    0.0001
    0.0001
    0.0000
```

This result would seem to indicate that $\lim_{n\to+\infty}(0.5)^n = 0$. In a similar manner, you can investigate the limiting behavior of the sequence generated by $(-0.75)^n$.

```
>> n=(1:30)';

>> (-0.75).^n

ans =

   -0.7500
    0.5625
   -0.4219
    0.3164
   -0.2373
    0.1780
   -0.1335
    0.1001
   -0.0751
    0.0563
   -0.0422
    0.0317
   -0.0238
    0.0178
   -0.0134
    0.0100
   -0.0075
    0.0056
   -0.0042
    0.0032
   -0.0024
    0.0018
   -0.0013
    0.0010
   -0.0008
    0.0006
   -0.0004
    0.0003
   -0.0002
    0.0002
```

Note that the sequence defined by $(-0.75)^n$ oscillates between positive and negative terms. The sequence defined by $(-0.75)^n$ converges to zero, though the convergence is much slower than the convergence of the sequence defined by $(0.5)^n$

Conjecture 1 *If λ is any real number such that $|\lambda| < 1$, then $\lim_{n\to+\infty} \lambda^n = 0$.*

Experiment. Using the Matlab technique demonstrated above, show that each of the following sequences converge to zero as $n \to +\infty$.

- $(0.25)^n$
- $(-.8)^n$
- $(.99)^n$

Conjecture 2 *If λ is any real number such that $|\lambda| > 1$, then the terms of the sequence generated by λ^n become infinitely large in magnitude.*

Experiment. Use Matlab to demonstrate that each of the following sequences produce terms of infinite magnitude as $n \to +\infty$.

- $(2.3)^n$
- $(-1.4)^n$
- $(1.05)^n$

1.2 When λ Is a Complex Number

If $\lambda = a + bi$, then the *magnitude* of λ is given by $|\lambda| = \sqrt{a^2 + b^2}$. For example, if $\lambda = 0.3 + 0.4i$, then the magnitude of λ is $|\lambda| = \sqrt{0.3^2 + 0.4^2} \approx .5$. Note in this case that $|\lambda| < 1$. Matlab easily computes the magnitude of a complex number.

```
>> abs(0.3+0.4i)

ans =

    0.5000
```

And the following Matlab code generates the first 15 terms of the sequence defined by $(0.3 + 0.4i)^n$.

```
>> n=(1:15)';
>> (0.3+0.4i).^n

ans =

   0.3000+ 0.4000i
```

```
  -0.0700+ 0.2400i
  -0.1170+ 0.0440i
  -0.0527- 0.0336i
  -0.0024- 0.0312i
   0.0118- 0.0103i
   0.0076+ 0.0016i
   0.0016+ 0.0035i
  -0.0009+ 0.0017i
  -0.0010+ 0.0001i
  -0.0003- 0.0003i
   0.0000- 0.0002i
   0.0001- 0.0001i
   0.0001+ 0.0000i
   0.0000+ 0.0000i
```

Note that the terms of the sequence generated by $(0.3+0.4i)^n$ converge to $0+0i$ (zero).

Conjecture 3 *If $|\lambda| < 1$, then the sequence defined by λ^n converges to zero.*

Experiment. Use Matlab to (a) show that the magnitude of λ is less than one, then (b) show that the sequence generated by λ^n converges to zero.

- $(0.25 + 0.45i)^n$
- $(-0.5 - 0.2i)^n$

Conjecture 4 *If $|\lambda| > 1$, then the terms of the sequence generated by λ^n become infinitely large in magnitude.*

For example, if $\lambda = 0.8 + 1.2i$, then $|\lambda| = \sqrt{0.8^2 + 1.2^2} \approx 1.4422$, which is larger than one.

```
>> abs(0.8+1.2i)

ans =

    1.4422
```

The following code will generate the first 15 terms of the sequence $(0.8+1.2i)^n$.

```
>> n=(0;15)';
>> S=(0.8+1.2i).^n

S =

  1.0e+002 *

   0.0100
```

```
  0.0080+ 0.0120i
 -0.0080+ 0.0192i
 -0.0294+ 0.0058i
 -0.0305- 0.0307i
  0.0125- 0.0611i
  0.0834- 0.0339i
  0.1074+ 0.0729i
 -0.0016+ 0.1872i
 -0.2259+ 0.1479i
 -0.3581- 0.1527i
 -0.1032- 0.5519i
  0.5798- 0.5654i
  1.1423+ 0.2434i
  0.6217+ 1.5654i
 -1.3812+ 1.9984i
```

And you can generate the magnitudes of these terms in the following manner.

```
>> abs(S)

ans =

    1.0000
    1.4422
    2.0800
    2.9998
    4.3264
    6.2396
    8.9989
   12.9784
   18.7177
   26.9951
   38.9329
   56.1498
   80.9804
  116.7916
  168.4393
  242.9266
```

Note that the magnitude of the terms of the sequence generated by $(0.8+1.2i)^n$ is increasing in size. The magnitude of the terms in this sequence will increase indefinitely[1].

Experiment. Use Matlab to (a) show that the magnitude of λ is larger than one, and (b) show that the terms of the sequence generated by λ^n become

[1]Check the validity of this statement by generating more than 15 terms of the sequence; maybe 25 terms.

infinite in magnitude.

- $(1.25 + 0.8i)^n$
- $(-1.4 - 0.8i)^n$

2 Difference Equations—Long Term Behavior

Consider the difference equation with initial condition defined by

$$\mathbf{x}_k = \begin{bmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{bmatrix} \mathbf{x}_{k-1}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1}$$

The eigenvalues and associated eigenvectors of matrix

$$A = \begin{bmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{bmatrix}$$

are

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = 1.2 \\ \mathbf{v}_2 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = .8 \end{aligned}$$

As you saw in "Difference Equations—Closed Form Solutions," if the initial condition can be written as a linear combination of the eigenvectors, as in $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then the closed form solution of equation (1) is given by

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 \tag{2}$$

In equation (2), because $|\lambda_1| > |\lambda_2|$ the eigenvalue λ_1 is called the *dominant eigenvalue*[2].

Divide both sides of equation (2) by the kth power of the dominant eigenvalue.

$$\begin{aligned} \frac{\mathbf{x}_k}{\lambda_1^k} &= \frac{c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2}{\lambda_1^k} \\ \frac{\mathbf{x}_k}{\lambda_1^k} &= c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 \end{aligned} \tag{3}$$

Now take the limit as $k \to +\infty$.

$$\begin{aligned} \lim_{k\to+\infty}\frac{\mathbf{x}_k}{\lambda_1^k} &= \lim_{k\to+\infty}\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2\right] \\ \lim_{k\to+\infty}\frac{\mathbf{x}_k}{\lambda_1^k} &= c_1\mathbf{v}_1 + c_2\lim_{k\to+\infty}\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 \end{aligned} \tag{4}$$

[2] If one eigenvalue is strictly larger in magnitude than all other eigenvalues, that eigenvalue is called the *dominant eigenvalue*.

However, because $|\lambda_1| > |\lambda_2|$ you know[3] that $|\lambda_2/\lambda_1| < 1$. Therefore, $\lim_{k\to+\infty}(\lambda_2/\lambda_1)^k = 0$ and

$$\lim_{k\to+\infty} \frac{\mathbf{x}_k}{\lambda_1^k} = c_1\mathbf{v}_1$$

Therefore, for large values of k,

$$\begin{aligned} \frac{\mathbf{x}_k}{\lambda_1^k} &\approx c_1\mathbf{v}_1 \\ \mathbf{x}_k &\approx c_1\lambda_1^k\mathbf{v}_1 \end{aligned} \tag{5}$$

Because c_1 and λ_1^k are scalars, equation (5) indicates that $\mathbf{x}_k$ is approximately a scalar multiple of $\mathbf{v}_1$ and must move approximately *parallel* to the eigenvector $\mathbf{v}_1$ as you iterate equation (1) forward in time.

3 The Trajectory Program

The Matlab M-file `traj2` can easily draw solutions of equation (1). Start the program by typing `traj2` at the Matlab prompt and enter the matrix in equation (1) when prompted.

```
>>traj2
Enter a 2x2 matrix in form [a,b;c,d] --> [1.0 0.2;0.2 1.0]
```

The program responds by creating a figure with an axes. Position your mouse at approximately the point (1,0)—the initial condition is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$—and click. A solution trajectory is drawn, first forward in time from your initial mouse position (initial condition $\mathbf{x}_0$), then backward in time. Note that this solution, shown in Figure 1, eventually moves parallel to the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Experiment. Create several more solution trajectories for equation (1) by clicking initial conditions $\mathbf{x}_0$ in the figure axes with your mouse. Note that as each of the trajectories move forward in time (the blue part of the trajectory) they eventually move parallel to $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

4 Homework

1. For each of the following difference equations (dynamical systems) perform each of the following tasks:

   (a) Use `poly`, `roots`, and the `null` command to find the eigenvalues and eigenvectors of the given matrix.

[3] If $|\lambda_1| > |\lambda_2|$, then $|\lambda_1|/|\lambda_1| > |\lambda_2|/|\lambda_1|$ or $1 > |\lambda_2|/|\lambda_1|$. Therefore, $|\lambda_2/\lambda_1| < 1$.

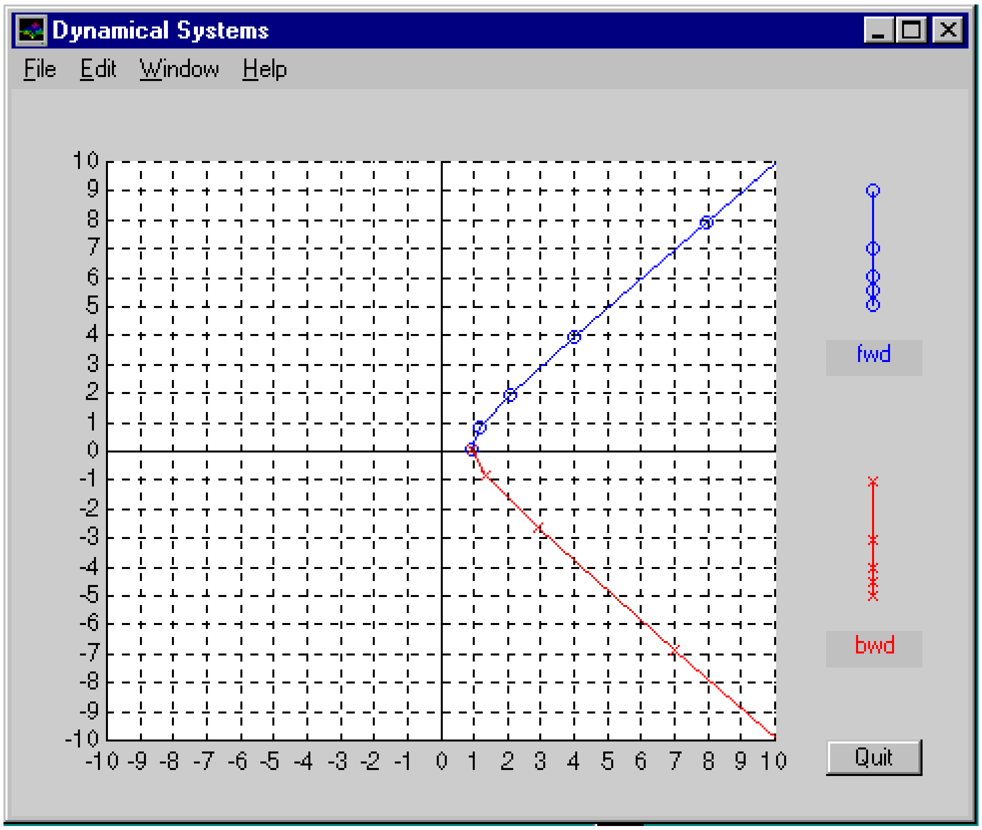

Figure 1: A solution trajectory of equation (1).

(b) Write the closed form solution of the difference equation in the form $\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2$.

(c) Divide both sides of the solution $\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2$ by the kth power of the dominant eigenvalue and take the limit as $k \to +\infty$. Use your result to approximate $\mathbf{x}_k$ for large values of k and predict the eventual behavior of the solution.

(d) Run the M-file `traj2` and show that the solution trajectories are behaving as you predicted in part (c).

(e) Obtain a printout for each problem. Use a ruler and a pencil to draw straight lines on your printed plot representing the eigenspace of each eigenvalue.

i. $\mathbf{x}_k = \begin{bmatrix} 0.6 & 0.2 \\ 0.0 & 0.8 \end{bmatrix}\mathbf{x}_{k-1}, \quad \mathbf{x}_0 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

ii. $\mathbf{x}_k = \begin{bmatrix} 1.42 & 0.16 \\ 0.16 & 1.18 \end{bmatrix}\mathbf{x}_{k-1}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$