Matlab and Coordinate Systems

Math 45 — Linear Algebra

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Abstract

In this exercise we will introduce the concept of a coordinate system for a vector space. The map from the vector space to its coordinate space is defined, then graph paper is crafted to accompany the corresponding basis. Moving from one coordinate system to another is explained, and the activity uses different kinds of graph paper to help the reader visualize the change of basis technique. *Prequisites: Knowledge of a basis for a vector space.*

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Introduction

This is an interactive document designed for online viewing. We've constructed this onscreen documents because we want to make a conscientious effort to cut down on the amount of paper wasted at the College. Consequently, printing of the onscreen document has been purposefully disabled. However, if you are extremely uncomfortable viewing documents onscreen, we have provided a print version. If you click on the Print Doc button, you will be transferred to the print version of the document, which you can print from your browser or the Acrobat Reader. We respectfully request that you only use this feature when you are at home. Help us to cut down on paper use at the College.

Much effort has been put into the design of the onscreen version so that you can comfortably navigate through the document. Most of the navigation tools are evident, but one particular feature warrants a bit of explanation. The section and subsection headings in the onscreen and print documents are interactive. If you click on any section or subsection header in the onscreen document, you will be transferred to an identical location in the print version of the document. If you are in the print version, you can make a return journey to the onscreen document by clicking on any section or subsection header in the print document.

Finally, the table of contents is also interactive. Clicking on an entry in the table of contents takes you directly to that section or subsection in the document.

Working with Matlab

This document is a working document. It is expected that you are sitting in front of a computer terminal where the Matlab software is installed. You are not supposed to read this document as if it were a short story. Rather, each time your are presented with a Matlab command, it is expected that you will enter the command, then hit the Enter key to execute the command and view the result. Furthermore, it is expected that you will ponder the result. Make sure that you completely understand why you got the result you did before you continue with the reading.

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The Coordinate Mapping

Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a *basis* for a vector space V over the real numbers R. Recall that this means that the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$ are *linearly independent*.

Definition 1

If the only linear combination of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$ equaling the zero vector is the trivial combination, then the vectors are *linearly independent*. That is, if

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$$

implies that $c_1 = c_2 = \cdots c_n = 0$, then the vectors are linearly independent.

Furthermore, recall that the basis vectors *span* the vector space. That is, each vector in V can be written as a linear combination of the basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$. Thus, if $\mathbf{v} \in V$, it is possible to find scalars c_1, c_2, ..., c_n in R such that

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

However, the scalars c_1, c_2, ..., c_n, when considered as a vector,

$$[\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

form a *coordinate vector* from R^n. When we think in this manner, we think of the scalars as the *coordinates* of the vector $\mathbf{v}$. We will consistently use the notation $[\mathbf{v}]_B$ to represent the coordinates of $\mathbf{v}$ with respect to the basis B.

It is natural at this point to define a transformation $T : V \to R^n$ by

$$\mathbf{v} \mapsto [\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

The difficulty lies in the fact that this transformation may not be well defined. That is, what if there is more than one way to express $\mathbf{v}$ as a linear combination of the basis vectors? Should we map the vector $\mathbf{v}$ to the first set of coordinates we find, or should we map the vector $\mathbf{v}$ to a second or third set of coordinates? Fortunately, as we shall see in the following lemma, this is not a possible scenario.

Lemma 1

Let V be a vector space with basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$. For every $\mathbf{v} \in V$, there is a *unique* set of scalars such that

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

Proof: Suppose that there is more than one way to express $\mathbf{v}$ as a linear combination of the basis elements. Then there exists scalars c_1, c_2, ..., c_n such that

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

and there exists scalars d_1, d_2, ..., d_n such that

$$\mathbf{v} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \cdots + d_n\mathbf{b}_n.$$

Subtract the second equation from the first to get

$$\mathbf{0} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Because the basis vectors are linearly independent, each of the coefficients in the last expression must equal zero. Therefore, $c_i = d_i$ for $i = 1, 2, \ldots, n$.

There is exactly one way that we can express a vector $\mathbf{v}$ as a linear combination of the basis vectors. Therefore, the transformation

$$T : \mathbf{v} \mapsto [\mathbf{v}]_B$$

is well defined. It is beyond the scope of this article, but you can also show that the transformation T is linear, onto, and one-to-one. This makes the vector space V *isomorphic* to the coordinate space R^n. The two spaces may look different, but they act in exactly the same manner. This is why it is so important that we learn as much about R^n as we possibly can.

Let's look at an example.

Example 1

Consider the vector space P_3, the space of all polynomials with real coefficients having degree less than 3. Are the vectors

$$\begin{aligned} p_1(x) &= 1 + x + x^2 \\ p_2(x) &= 1 - 2x \\ p_3(x) &= -2 + 13x + 3x^2 \end{aligned}$$

linearly independent?

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This is somewhat tricky to show in the usual manner. However, it is not difficult to show that $B = \{1, x, x^2\}$ is a basis for P_3. With this basis, the coordinate transformation makes the following assignments:

$$[p_1(x)]_B \mapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$[p_2(x)]_B \mapsto \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

$$[p_3(x)]_B \mapsto \begin{pmatrix} -2 \\ 13 \\ 3 \end{pmatrix}.$$

Because the space P_3 is isomorphic to R^3, the polynomials will be independent if and only if we can say the same thing about their coordinate representations. Can we find scalars, not all zero, such that

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 13 \\ 3 \end{pmatrix} = \mathbf{0}.$$

To solve this equation, first setup the augmented matrix in Matlab.

```
>> M=[1 1 -2 0;1 -2 13 0;1 0 3 0]
M =
     1     1    -2     0
     1    -2    13     0
     1     0     3     0
```

Row reduce.

```
>> rref(M)
ans =
      1     0     3     0
      0     1    -5     0
      0     0     0     0
```

Thus, $c_1 = -3c_3$, $c_2 = 5c_3$, and c_3 is free. Letting $c_3 = 1$, one possible solution is $c_1 = -3$, $c_2 = 5$, and $c_3 = 1$. Thus, the coordinate vectors are dependent because

$$-3\begin{pmatrix}1\\1\\3\end{pmatrix} + 5\begin{pmatrix}1\\-2\\0\end{pmatrix} + 1\begin{pmatrix}-2\\13\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Because the space P_3 is isomorphic to R^3, the same dependency will exist amongst the polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$.

$$\begin{aligned}-3p_1(x) + 5p_2(x) + 1p_3(x) &= -3(1 + x + x^2) + 5(1 - 2x) + 1(-2 + 13x + 3x^2)\\ &= 0 + 0x + 0x^2\end{aligned}$$

Graph Paper

The idea of a coordinate transformation might seem new and quite abstract. However, it is something that you have always dealt with ever since you began your studies in mathematics. Take for example, the Cartesian plane, where every point in the plane is associated with a pair of coordinates, and similarly, every pair of coordinates is associated with a point in the plane. This is precisely the transformation

$$\mathbf{v} \mapsto [\mathbf{v}]_B$$

that we are talking about. Every vector $\mathbf{v}$ has a coordinate representation $[\mathbf{v}]_B$ and every set of coordinates represents a vector in the vector space.

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The "standard" basis for the plane is

$$E = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Note that $\mathbf{e}_1$ and $\mathbf{e}_2$ are the first and second columns of the 2×2 identity matrix, respectively. Every point in the plane can be written as a unique linear combination of these two basis vectors. For example, the point $(2, 4)$ is written

$$\begin{aligned} \begin{pmatrix} 2 \\ 4 \end{pmatrix} &= 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &= 2\mathbf{e}_1 + 4\mathbf{e}_2. \end{aligned}$$

Indeed, when we speak of the "point" $(2, 4)$, we are actually presenting the coordinates of the point, and not the point itself. That's how confident we are that the plane is isomorphic to R^2. We speak of them as if they were identical.

Once we declare a basis for the plane, a corresponding coordinate system is automatically created. We usually superimpose this coordinate system on the plane by means of a grid, as shown in **Figure 1**.

It is important to understand the role of the basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

when constructing the graph paper. If a point in the plane has coordinates $(2, 4)$, then the position of this point is attained by traveling 2 units in the $\mathbf{e}_1$ direction, followed by 4 units in the $\mathbf{e}_2$ direction. Consequently, the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are fundamental in determining the look and feel of the grid.

However, as we know, there are many different bases we can use for the plane. Consider, for example, the basis

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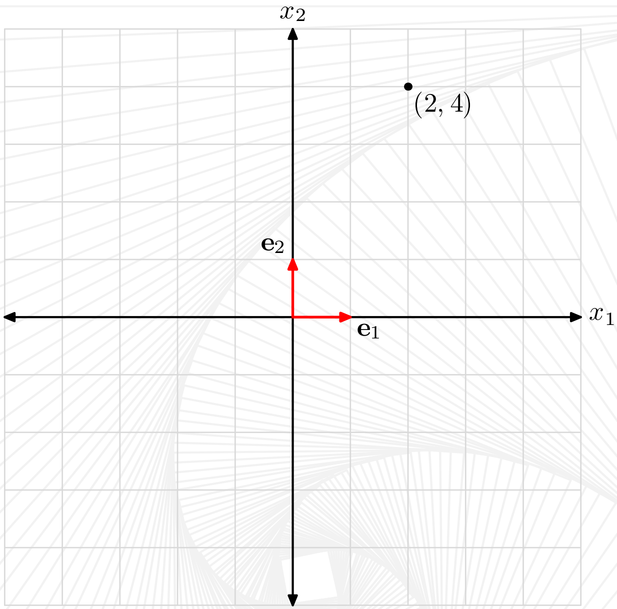

Figure 1 The standard coordinate system for the plane.

$$C = \{\mathbf{c}_1, \mathbf{c}_2\}, = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

which, as can be seen in **Figure 2**, creates an entirely different looking graph paper.

Even though our point is located in the same position in the plane, it now has a different set of coordinates. Indeed, the C-coordinates of the point are $(3, 1)$, which indicate that we must travel 3 units in the $\mathbf{c}_1$ direction, followed by 1 unit in the $\mathbf{c}_2$ direction.

That we have arrived at precisely the same point as we did in **Figure 1** is easy to see, as

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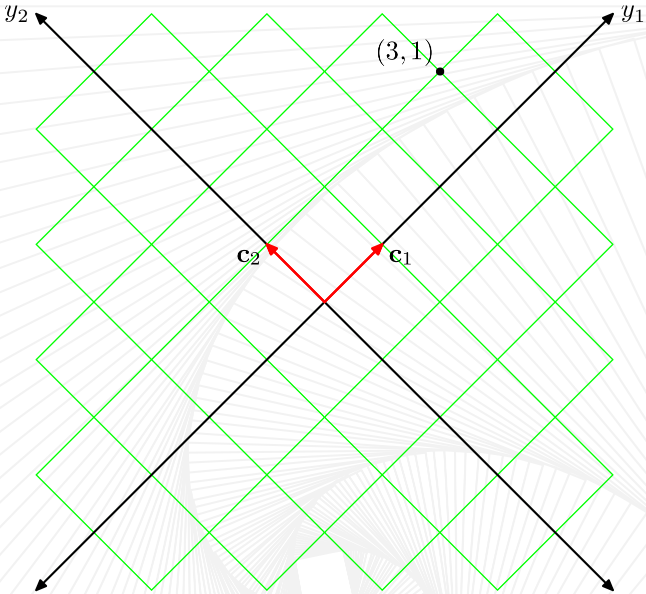

Figure 2 A second coordinate system for the plane.

$$3\mathbf{c}_1 + 1\mathbf{c}_2 = 3\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}-1\\1\end{pmatrix},$$
$$= \begin{pmatrix}2\\4\end{pmatrix},$$

which are the E-coordinates of the point. But this is even easier to see if we superimpose one coordinate system on another, as shown in Figure 3.

Change of Coordinates Matrix

Again, let

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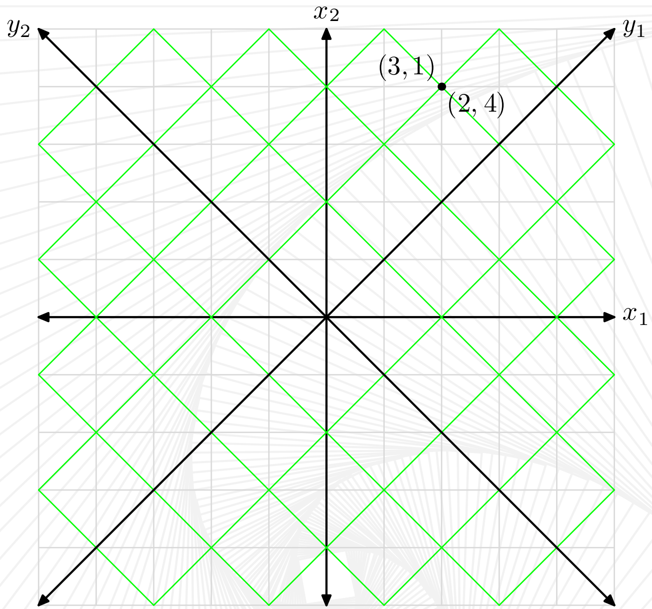

Figure 3 Combined coordinates systems for the plane.

$$C = \{\mathbf{c}_1, \mathbf{c}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

be the basis for the plane shown in **Figure 2**. How does one go about finding the *E*-coordinates of the point shown in **Figure 1**, given the *C*-coordinates of the point in **Figure 2**?

Because the point has *C*-coordinates

$$[\mathbf{x}]_C = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

we may write

$$\mathbf{x} = 3\mathbf{c}_1 + 1\mathbf{c}_1 = 3\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

Of course, the same task is easily accomplished with matrix multiplication.

$$\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

Note that the columns of the matrix are the C-basis vectors. Let P_C represent the matrix that transforms C-coordinates into "standard" E-coordinates.

$$P_C = [\mathbf{c}_1\ \mathbf{c}_2] = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$$

Thus, if you are given the C-coordinates of the vector $\mathbf{x}$, then you can find the E-coordinates in the "standard" basis as follows:

$$[\mathbf{x}]_E = P_C[\mathbf{x}]_C.$$

Now, because the C-basis vectors are independent, the matrix P_C is invertible. Consequently, given the "standard" E-coordinates of a point, one can find the C-coordinates with the following computation.

$$P_C^{-1}[\mathbf{x}]_E = [\mathbf{x}]_C$$

Example 2

Given $[\mathbf{x}]_C = \begin{pmatrix}-1\\-3\end{pmatrix}$, find $[\mathbf{x}]_E$.

A simple multiplication by the change of coordinates matrix P_C does the trick.

$$[\mathbf{x}]_E = P_C[\mathbf{x}]_C = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = \begin{pmatrix}2\\-4\end{pmatrix}$$

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Matlab can be extremely helpful when making coordinate transformations.

```
>> x_C=[-1;-3]
x_C =
    -1
    -3
>> P_C=[1 -1;1 1]
P_C =
     1    -1
     1     1
» x_E=P_C*x_C
x_E =
     2
    -4
```

Of course, abstract computations can be quite meaningless unless accompanied by some sort of visualization. When you plot the coordinates given by $[\mathbf{x}]_E$ on E-coordinate graph paper, then plot the coordinates given by $[\mathbf{x}]_C$ on C-coordinate graph paper, things become clear, especially when you superimpose one coordinate system atop the second. This is shown in **Figure 4**.

Example 3

Given $[\mathbf{x}]_E = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $[\mathbf{x}]_C$.

Again, the change of coordinates matrix provides the answer.

$$[\mathbf{x}]_C = P_C^{-1}[\mathbf{x}]_E = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Matlab eases the computations.

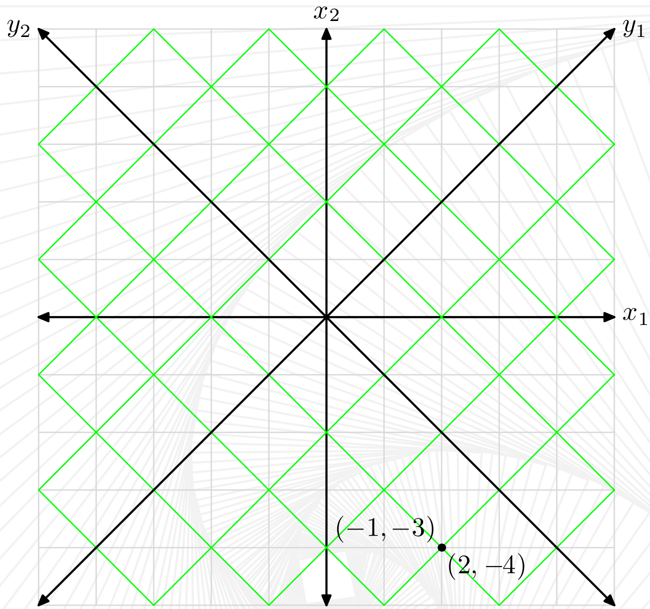

Figure 4 The coordinates of the point in the C-coordinate system are given by $[\mathbf{x}]_C = (-1, -3)$. The E-coordinates are $[\mathbf{x}]_E = (2, -4)$.

```
>> x_E=[-3;1]
x_E =
    -3
     1
>> x_C=inv(P_C)*x_E
x_C =
    -1
     2
```

Again, this is intractable without visualization. The coordinates $[\mathbf{x}]_E$ and $[\mathbf{x}]_C$ are plotted in the appropriate coordinate systems in **Figure 5**.

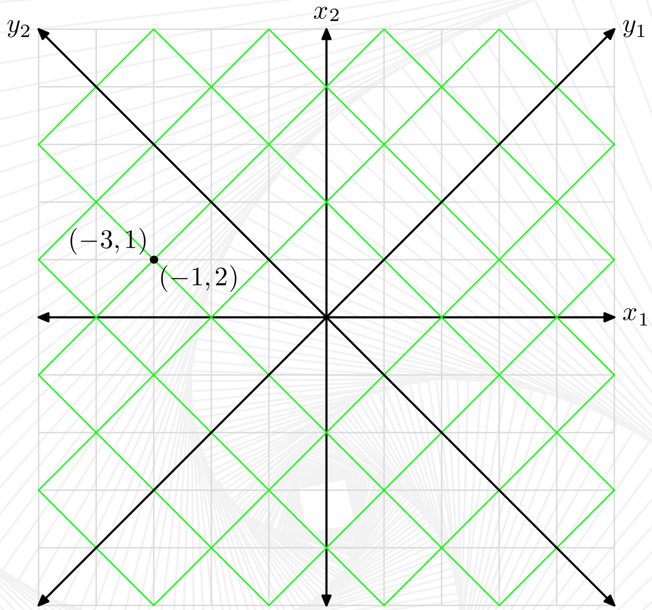

Figure 5 The coordinates of the point in the C-coordinate system are given by $[\mathbf{x}]_C = (-1,2)$. The E-coordinates are $[\mathbf{x}]_E = (-3,1)$.

Homework

Recall the definition of a basis.

Definition 2

Let V be a vector space over R. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a collection of vectors from V. The set B is a *basis* for V is and only if two conditions are satisfied:

- The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are linearly independent.
- The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ span the space V. That is, any $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$.

Thus, to show that

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

is a basis for R^2, you need to show two things:

- The vectors $[2, 1]^T$ and $[-1, 1]^T$ are linearly independent. That is, if
$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
then $c_1 = c_2 = 0$.
- The vectors $[2, 1]^T$ and $[-1, 1]^T$ span R^2. That is, for any $[a, b]^T \in R^2$, there exists scalars c_1 and c_2 such that
$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

These ideas will be important as you attempt to complete the following exercises for homework.

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1. Consider the collection of vectors from R^2,

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

a. Prove that B is a basis for R^2.

b. On a sheet of "standard" graph paper, superimpose a new grid representing the B-coordinate system.

c. Plot each of the following coordinates on your B-coordinate graph paper.

i. $[\mathbf{x}]_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

ii. $[\mathbf{x}]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

iii. $[\mathbf{x}]_B = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

iv. $[\mathbf{x}]_B = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

d. Use your graph paper to determine the "standard" E-coordinates of each point plotted in part (c).

e. The change of coordinate matrix is

$$P_B = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \tag{1}$$

where the columns of P_B are the basis vectors of B, with $[2, 1]^T$ appearing first and $[-1, 1]$ appearing second (order matters). Use the relation

$$[\mathbf{x}]_E = P_B[\mathbf{x}]_B$$

to change each B-coordinate in part (c) to "standard" E-coordinates. Verify that each result agrees with the visual result obtained in part (d).

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f. Craft a second sheet of B-coordinate graph paper, superimposing a B-coordinate grid on a "standard" sheet of E-coordinate graph paper. Plot each of following the "standard" E-coordinate vectors on your graph paper.

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i. $[\mathbf{x}]_E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ii. $[\mathbf{x}]_E = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

iii. $[\mathbf{x}]_E = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

iv. $[\mathbf{x}]_E = \begin{pmatrix} -12 \\ -3 \end{pmatrix}$

g. Use your graph paper to determine the B-coordinates of each point plotted in part (f).

h. Use the relation

$$[\mathbf{x}]_B = P_B^{-1}[\mathbf{x}]_E$$

to determine the B-coordinates of the each "standard" E-coordinate in part (f). Verify that each result agrees with the visual result obtained in part (g).

2. Here is what it means for a function to map a space *onto* another space.

Definition 3

A function T mapping a vector space X into a vector space Y is said to be *onto* Y if and only if for each $\mathbf{y} \in Y$ there exists an $\mathbf{x} \in X$ such that $T(\mathbf{x}) = \mathbf{y}$.

Let V be a vector space over R with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Define $T : V \mapsto R^n$ by

$$T(\mathbf{v}) = [\mathbf{v}]_B.$$

Show that the mapping T is onto R^n.

3. Here is what it means for a function to be *one-to-one*.

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Definition 4

Let T map the vector space X into the vector space Y. The mapping T is one-to-one if and only if different vectors get mapped to different vectors. That is, if $\mathbf{x} \neq \mathbf{y}$, then $T(\mathbf{x}) \neq T(\mathbf{y})$.

Let V be a vector space over R with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Define $T : V \mapsto R^n$ by

$$T(\mathbf{v}) = [\mathbf{v}]_B.$$

Show that the mapping T is one-to-one.