# Linear Algebra and *Azulejos*

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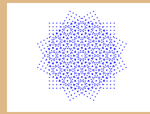
**Abstract**

The purpose of this project is to explain the use of Linear Algebra in *azulejo* tile design.

## Introduction

A unique application of Linear Algebra involves applying techniques to visual phenomena. A little known application explained by Federico Fernández involves creating pleasing designs for use with *azulejos*, a Spanish art of tile making. The application of his technique extends much further than tiles, the designs themselves are quite tasteful but involve a great deal of mathematics.





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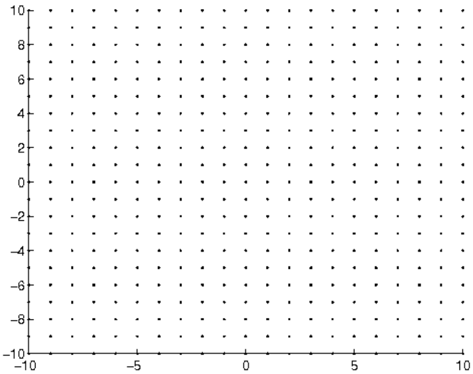


Figure 1: A simple lattice.

Federico Fernández has created what he calls a "semiautomatic" way to create these designs. The purpose of this piece is to elaborate on this process and in turn create our own *azulejos*. We will begin with defining a lattice as all integer linear combinations of a pair of vectors. As you can see in Figure 1, the simple basis for a lattice containing all ineger coordinates is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

When choosing vectors as your basis be certain that its determinant has an absolute value of 1, a much more detailed explanation for this rule

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will soon follow, but before we get there we must begin an explanation of our lattices.

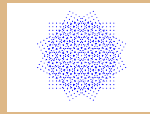
The individual markers defined in the lattice can each be expressed as linear combinations of the natural numbers of our two vectors.

# Nets of Class $n$

If two vectors are taken as: $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} c \\ d \end{bmatrix}$ and each are linearly independent from the other and contain integer entries, then the "group" $\Lambda$ contains all integer linear combinations of these vectors as $m\mathbf{v}+n\mathbf{w}$ and will be called a sublattice with $\mathbf{v}$ and $\mathbf{w}$ as the basis for the integer lattice $\Lambda_0$. The resulting parallelogram in $\mathbb{R}^2$ consists of points $s\mathbf{v} + t\mathbf{w}$ where $0 \leq s$ and $t \leq 1$. We will call this arrangement the *fundamental parallelogram* for the basis. With knowledge of determinants one can find the area of this parallelogram as the determinant of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \Lambda$ then the equation $\mathbf{x} = m\mathbf{v} + n\mathbf{w}$ can be expressed in matrix form as:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}$$

This shows that the points of the sublattice $\Lambda$ are from multiples of the basis which has been defined as $A$ and elements of the integer lattice $\Lambda_0$.



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Using our defined variables for cleanliness this is equivalent to $\Lambda = A\Lambda_0$. We will elaborate more on *nets of class n* shortly.

# Unimodularity

Let's begin by defining $\mathbb{Z}_{2x2}$ as all 2 x 2 matrices which contain only integer entries. There will be special circumstances where a matrix $U$ (a *unimodular* matrix) if $U \in \mathbb{Z}_{2x2}$ and the determinant of $U$ is equal to $\pm 1$.

## Any unimodular matrix $U$ is a basis for $\Lambda_0$; in other words $U\Lambda_0 = \Lambda_0$

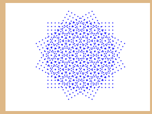
If we have a set of two vectors defined in $U$ and these vectors are chosen so that their determinant is $\pm 1$ then,

$$U^{-1} = \frac{1}{\pm 1} \cdot \begin{bmatrix} u_4 & -u_2 \\ -u_3 & u_1 \end{bmatrix}$$

This leaves $U^{-1}$ as having integer coefficients and also creating a situation where $U \cdot U^{-1} = I$

Take for example

$$U = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$





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The columns of the Identity matrix can be written as a linear combination of the columns of our basis vectors:

$$x = m_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + n_1 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$y = m_2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + n_2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x + y = (m_1 + m_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (n_1 + n_2) \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \left( \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

The basis of $\Lambda_0$ are integer linear combinations of the columns of $U$ where the coefficients are entries in $U^{-1}$. Therefore the sublattice generated by $U$ is the integer lattice of $\Lambda_0$.

For emphasis if $U = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then $U^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. We will then receive,

$$
\mathbf{e_1} = 3\begin{bmatrix}2\\1\end{bmatrix} - 1\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}
$$

$$
\mathbf{e_2} = -5\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}
$$

which shows that $U\Lambda_0 = \Lambda_0$, demonstrating that the basis of $\Lambda_0$ can be written with a unimodular matrix $U$.

## Multiple matrices can define $\Lambda$.

In the case where we have two matrices $A$ and $B$ in $\mathbb{Z}_{2x2}$, where their bases can be the same sublattice $\Lambda$, we will see a unique relationship. This will only occur if there exists a unimodular matrix $U$ so that $A = BU$ or in other words $B^{-1}A$ is unimodular.

$$
\begin{aligned}
A &= BU \\
A\Lambda_0 &= (BU)\Lambda_0 \\
&= B(U\Lambda_0) \\
&= B\Lambda_0
\end{aligned}
$$

This shows that $B$ and $A$ are bases for the same sublattice $\Lambda$. Merely for thoroughness if we once again have an $A$ and $B$ in $\mathbb{Z}_{2x2}$ but know that

$A$ and $B$ are in fact bases for the same sublattice, where $A = BU$ when $U \in \mathbb{Z}_{2x2}$ and $B = AV$ where $V \in \mathbb{Z}_{2x2}$ then by substituting we get,

$$A = AVU$$

And if we take the determinants

$$|A| = |A||V||U|$$

When this is done we see that $|V||U| = 1$ because we have defined unimodular matrices as having a determinant of $\pm 1$ and if these unimodular matrices are defining the same sublattice then their determinant values will be equal (i.e. $|V| = 1$ then $|U| = 1$ or $|V| = -1$ then $|U| = -1$). Because the determinants are equal and they exist with such constraints that their determinant will always be $\pm 1$ the product will be 1. Due to this we see that it is $U$ that must be unimodular in this case. We will now refer to a sublattice $\Lambda_0$ with a basis matrix whose determinant is a natural $n$ as a *net of class* $n$.

# The Number of Distinct Nets of Class $n$

Each natural number has exactly $\sigma(n)$ distinct nets of class $n$. Every net $\Lambda$ of class $n$ has a $\sigma(n)$ matrix in the form of $\begin{bmatrix} d & k \\ 0 & \frac{n}{d} \end{bmatrix}$ as a basis where $d$ divides $n$ with constraints $0 \le k \le d-1$. This matrix will be the canonical basis for $\Lambda$.

Let $A = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$. If we take the determinant of A we receive $|A| = xw - yz = n$. This can be represented in matrix form as:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = w \begin{bmatrix} x \\ y \end{bmatrix} - y \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} wx - yz \\ wy - yw \end{bmatrix}$$

Now if we supposed that $g$ divides $w$ and that it can also divide $y$ then $g$ can also divide $n$ and would appear in this form $\begin{bmatrix} \frac{n}{g} \\ 0 \end{bmatrix}$ with this point being within the lattice.

$$wx - yz = n \Rightarrow g|n \text{ let } a = \frac{w}{g} \text{ and } b = \frac{y}{g} \text{ then,}$$

$$a\begin{bmatrix}x\\y\end{bmatrix} - b\begin{bmatrix}z\\w\end{bmatrix} = \begin{bmatrix}ax-bz\\ay-bw\end{bmatrix}$$
$$= \begin{bmatrix}\frac{wx}{g}-\frac{yz}{g}\\ \frac{wy}{g}-\frac{yw}{g}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{wx-yz}{g}\\ 0\end{bmatrix}$$
$$= \begin{bmatrix}\frac{n}{g}\\ 0\end{bmatrix}$$

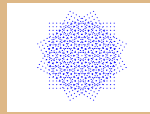
To continue this proof we need to let $\begin{bmatrix}d\\0\end{bmatrix}$ be the first grid point on the positive x-axis. Then $d = px - qz$ and $0 = py - qw$ for some $p$ and $q$. This is from the idea that,

$$\begin{bmatrix}d\\0\end{bmatrix} = p\begin{bmatrix}x\\y\end{bmatrix} - q\begin{bmatrix}z\\w\end{bmatrix}$$

Therefore,

$$\begin{bmatrix}x & z\\ y & w\end{bmatrix}\begin{bmatrix}p\\-q\end{bmatrix} = \begin{bmatrix}d\\0\end{bmatrix}$$

We notice that $p$ and $q$ must be relatively prime or otherwise there would be smaller lattice points. By Cramer's Rule we note $np = dw$ and $nq = dy$.



$$p = \frac{\left|\begin{bmatrix} d & z \\ 0 & w \end{bmatrix}\right|}{n} = \frac{dw}{n}$$

$$q = \frac{\left|\begin{bmatrix} x & d \\ y & 0 \end{bmatrix}\right|}{n} = \frac{-yd}{n}$$

Then $\begin{bmatrix} p \\ q \end{bmatrix} = 1 \Rightarrow d|n$. This is also useful as we can now call $\begin{bmatrix} d & k \\ 0 & \frac{n}{d} \end{bmatrix}$ the canonical basis for $\Lambda$.

# Primitive Points

A primitive point $\begin{bmatrix} a & b \end{bmatrix}$ is a point in $\Lambda$, a sublattice of $\Lambda_0$, if the line segment from the origin to $\begin{bmatrix} a & b \end{bmatrix}$ in $\mathbb{R}^2$ contains no other points.

## Any primitive point of $\Lambda$ can be extended to a basis for $\Lambda$.

If we take $\begin{bmatrix} a & b \end{bmatrix}$ to be a primitive point and take $\begin{bmatrix} c & d \end{bmatrix}$ as a point that is as close to the origin and $\begin{bmatrix} a & b \end{bmatrix}$ as $\Lambda_0$ allows but not on the origin or vector defining $\begin{bmatrix} a & b \end{bmatrix}$ then $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$ must form a basis for $\Lambda$.

Suppose there is a point $\begin{bmatrix} x & y \end{bmatrix}$ in $\mathbb{R}^2$ but is not a linear combination of integers of $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$, then $\begin{bmatrix} x & y \end{bmatrix}$ is not on a vertex of the sublattice of $\Lambda_0$, as all vertex points of $\Lambda_0$ are from integer linear combinations of $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$ which span $\Lambda_0$.

Therefore, if we subtract enough multiples of these two vectors from $\begin{bmatrix} x & y \end{bmatrix}$ we will arrive at a point which would be in the area defined by the parallelogram of $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$ but still is not a vertex of our lattice. Now this new point $\begin{bmatrix} x' & y' \end{bmatrix}$ cannot be closer to the line defined by $\begin{bmatrix} a & b \end{bmatrix}$ because $\begin{bmatrix} c & d \end{bmatrix}$ has been defined as the closest possible lattice point defined by a vector. So $\begin{bmatrix} x' & y' \end{bmatrix}$ must be on the edge defined by $\begin{bmatrix} c & d \end{bmatrix}$ and $\begin{bmatrix} a+c & b+d \end{bmatrix}$, but if this were the case $\begin{bmatrix} x'-c & y'-d \end{bmatrix}$ would be a vertex contradicting the statement that $\begin{bmatrix} a & b \end{bmatrix}$ is a primitive point in $\Lambda$.

# Lattices Are A Group

Begin with $\Lambda_0$ as the Integer Lattice.

Show $\Lambda_0$ is a group under addition

$$\forall\; x, y \in \Lambda_0$$
$$x + y \in \Lambda_0$$

With natural numbers addition to another element will result in a natural number.



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$$x + (y + z) = (x + y) + z$$

The grouping will not affect the outcome or force the sum to leave $\Lambda_0$. There also exists a zero element.

$$x + 0 = x$$

There also exists an inverse element which will return zero.

$$x + (-x) = 0$$

# Symmetry of Net Classes

Analogous to Abstract Algebra concepts there will exist groupings of these subgroups classified as "types." These types are defined using the symmetry of a square. Any operation which results in the same orientation of the square will be of one type. There are eight types of symmetry in a square; namely symmetry across $y = 0$, $x = 0$, $y = x$, and $y = -x$ and four rotational symmetries.

To elaborate, if we have a matrix Q of an orthogonal transformation and has orthogonal column vectors then

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$$Q = \begin{bmatrix} p & q \\ \pm\sqrt{1-p^2} & \pm\sqrt{1-q^2} \end{bmatrix} \text{ Where } p \text{ and } q \text{ must be integers.}$$

The rows of Q are also orthogonal $(p^2 + q^2 = 1)$ so to keep this valid we must have $p = \pm 1$ and $q = 0$ or $p = 0$ and $q = \pm 1$. Due to this restriction we only have eight possible orthogonal 2 x 2 matrices with these integer entries, these being:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \alpha\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

additionally their negatives will be the other four possibilities. Once again this class of matrices are all recognized by their conformity to the symmetry to a square. For example, $\alpha^2 = -I$ which places all defined lattice points directly upon the previously defined lattice points; which results in no change, and is completely useless for creating interesting designs. This is shown with $\begin{bmatrix} x \\ y \end{bmatrix} \in \Lambda \Rightarrow \begin{bmatrix} -x \\ -y \end{bmatrix} \in \Lambda$. We can arrive to this conclusion with three subgroups:

$$\begin{aligned} \langle \alpha \rangle &= \{I, \alpha, \alpha^2 = -I, \alpha^3 = -\alpha\} \\ \langle \alpha^2, \beta \rangle &= \{I, -I, \beta, -\beta\} \\ \langle \alpha^2, \alpha\beta \rangle &= \{I, -I, \alpha\beta, -\alpha\}, \end{aligned}$$

where elements on the left are what create this situation and those on the right are the members of the subgroup.

This is where Federico Fernández's genius takes light. His solution for creating these pleasing designs involves these concepts. His "semiautomatic" method involves choosing a suitable net $\Lambda$ and superimposing the four nets of $\Lambda$, $\alpha(\Lambda)$, $\beta(\Lambda)$, and $\alpha\beta(\Lambda)$ which are nets of the same type and are distinct. Easily enough if $\Lambda$ is fixed by either $\alpha$, $\beta$, or $\alpha\beta$ but is not linked to the others we will only receive two distinct nets, but if $\Lambda$ is fixed by *both* $\alpha$ and $\beta$ (which automatically makes $\alpha\beta$ fixed) then all four of these nets will be identical.

If we have a basis of $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ then we are always fixed by $\alpha\beta$ through the reflection line $y = 0$ and is sent to the new basis of $\begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}$. So these nets will always be of type $R$ due to the results being rectangular with horizontal and vertical bases, for example Figure 2.

The net with a basis $\begin{bmatrix} n & 1 \\ 0 & 1 \end{bmatrix}$ is fixed by $\beta$ and is sent to a new basis of $\begin{bmatrix} n & n-1 \\ 0 & 1 \end{bmatrix}$ by $\alpha$ and $\alpha\beta$. These two nets form a second type, we will call this type $D$ (Refer to Figure 3 due to the type being defined by diagonal vectors: $\begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} n & 0 \end{bmatrix} - \begin{bmatrix} n-1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$.

Nets of class $n$ always will contain the two degenerate types $R$ and $D$. There will be other degenerate types.

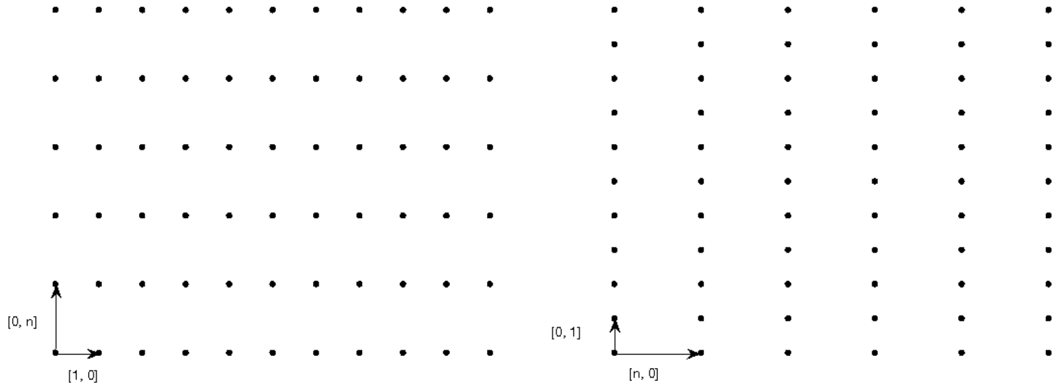


Figure 2: Examples of R Basis Setup

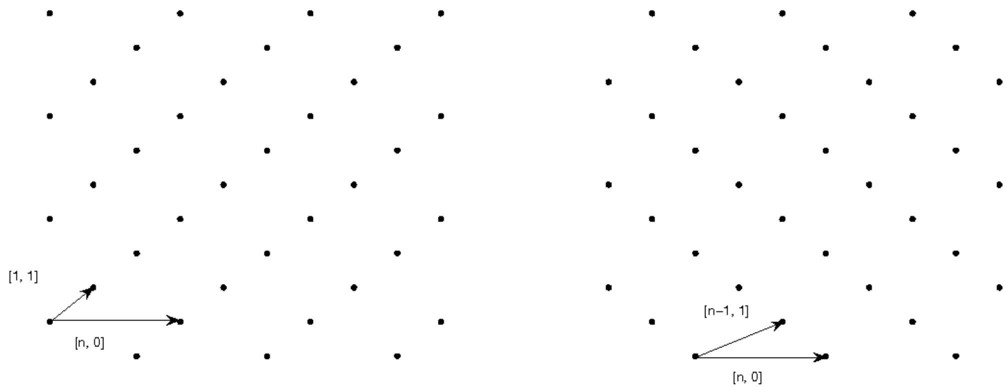


Figure 3: A Typical D Basis

## Degenerate Types of $\sigma(n)$

For our purposes $\sigma(p)$ represents the sum of the divisors of $p$ where $p$ is an odd prime number greater than 2. For example:

$$\begin{aligned}
\sigma(3) &= 1 + 3 = 4 \\
\sigma(5) &= 1 + 5 = 6 \\
\sigma(7) &= 1 + 7 = 8 \\
&\vdots \\
\sigma(p) &= p + 1
\end{aligned}$$

If we let $p$ be an odd prime number in $\sigma(p) = p + 1$. For example $p \equiv 3 \pmod 4$ so excluding types $R$ and $D$ (each contain two nets) then all the others contain four nets. It is important to again note that if the four nets of a type are superimposed one can expect an intricate design.

Modular Linear Algebra can best be summed with a number line wrapped around a circle of four markers (in our case) This is modeled with Figure 4. We have four possibilities of results and continuing the analogy we see that 5, 9, . . . can serve the same purposes as 1. For our application we have a total of eight symmetries, four are merely the negatives of the others. So we truly have four possibilities and notation such as:

$$19 \equiv 3(mod\ 4)$$



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means that there are four possibilities (mod 4) and after the last full revolution around the circle you end up on the value 19 which is equivalent to 3.

If a net $\Lambda$ has a basis $\begin{bmatrix} p & k \\ 0 & 1 \end{bmatrix}$ with $1 \leq k \leq p-1$ it cannot be fixed by $\alpha\beta$ (recall that $\alpha\beta$ will reflect across $y = x$ and reflect once more leaving the original net the same).

$$\alpha\beta\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & k \\ 0 & -1 \end{bmatrix}$$

If $\begin{bmatrix} p & k \\ 0 & -1 \end{bmatrix}$ were a basis of $\Lambda$ we would need the linear combinations to have the capability to define all lattice points of our net (when integers are used).

$$\varrho \begin{bmatrix} p \\ 0 \end{bmatrix} + \varsigma \begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ -1 \end{bmatrix}$$

When we solve for the equations we receive $\varrho p + \varsigma k = k$ and $\varsigma = -1$. Plugging $\varsigma$ into the first equation we see $\varrho p = 2k$. From our earlier explanation on abstract algebra we know that for this to be true we will need $2k \equiv 0 (\bmod p)$ but this proves impossible as $1 \leq k \leq p-1$. So any prime number will have $p+1$ nets of class $p$ and will have two or four distinct nets.

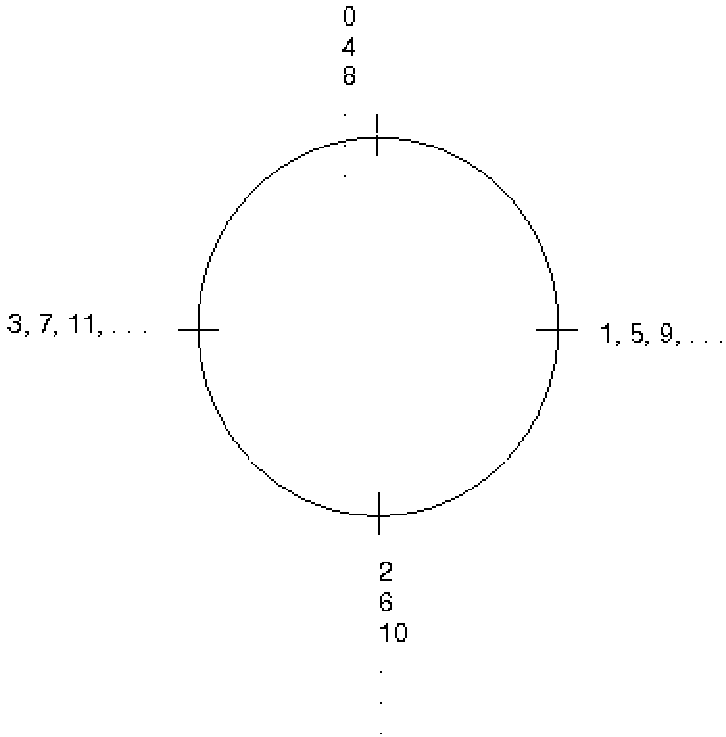



Figure 4: Dave's Picture for Abstract Algebra.

Closely related to this we can see that $\Lambda$ will be fixed by $\beta$ only if $k^2 \equiv 1 (\text{mod } p)$. This being the case then $k = 1$ or $k = p - 1$, thus $\Lambda$ will be a net of type $D$.

Additionally if $\Lambda$ is fixed by $\alpha$ we will then work with

$$\alpha\Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -p & -k \end{bmatrix}$$

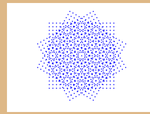
Which results in:

$$k \begin{bmatrix} p \\ 0 \end{bmatrix} - p \begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -p \end{bmatrix}$$

If $\begin{bmatrix} 0 & 1 \\ -p & -k \end{bmatrix}$ were to be a basis of $\Lambda$ we must have:

$$\varrho \begin{bmatrix} p \\ 0 \end{bmatrix} + \varsigma \begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -k \end{bmatrix}$$

Following the same process as demonstrated earlier we find $\varrho p - k^2 = 1$ and $k^2 \equiv -1 (\text{mod } p)$ but this creates a dilemma as a squared unknown is taking on a negative value. We can see for example in Figure 5 undergoing this process where several superpositions are redundant and it creates a very simple design, simple in terms of what is possible with the "semiautomatic" process.



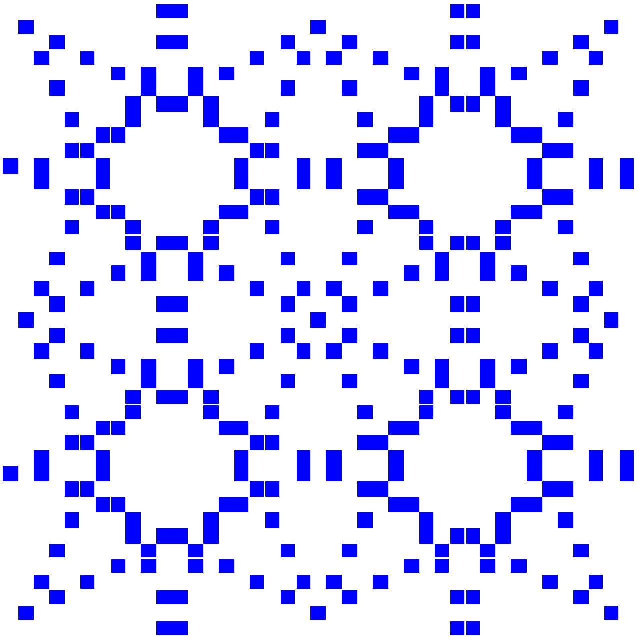


Figure 5: A Degenerate Basis.

This is an example of a lattice with canonical bases $\begin{bmatrix} 19 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 19 & 17 \\ 0 & 1 \end{bmatrix}$ put under the influence of the earlier stated matrices $\alpha$, $\beta$, and $\alpha\beta$. The result could have just as easily been created with merely multiplying $\alpha$ against our canonical bases so $\beta$ and $\alpha\beta$ have added no new information.

So now we can see that $p \equiv 3 \pmod 4$ of $R$ and $D$ will only produce two nets. For our $p \equiv 1 \pmod 4$ there are three types that will return two nets:

$$
\begin{aligned}
R &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \right\}, \\
D &= \left\{ \begin{bmatrix} p & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & p-1 \\ 0 & 1 \end{bmatrix} \right\}, \\
&\text{and } \left\{ \begin{bmatrix} p & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & p-k \\ 0 & 1 \end{bmatrix} \right\} \\
&\text{where } k^2 \equiv -1 \pmod p
\end{aligned}
$$

These degenerate cases will always produce a simple array of two nets but under normal circumstances we will receive four distinct nets which will prove very pleasing to a viewer when their results are superimposed upon one another. For example if we slightly modify the earlier canonical bases to conform to our guidelines we will receive: $\begin{bmatrix} 19 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 19 & 7 \\ 0 & 1 \end{bmatrix}$. When these bases are put against $\alpha$, $\beta$, and $\alpha\beta$ the result changes dramatically. In fact Figure 6 is exactly the same as Federico's prime example.

## Perfecting The *Azulejo*

As one can see the *azulejo* tiles do not have to be drawn with dots, even the shape and size of your marker will affect your results. Experimentation with color and and shape will prove to be satisfying next steps with the semiautomatic process. Once someone becomes fluent in this process great designs can flourish, each mathematically perfect.

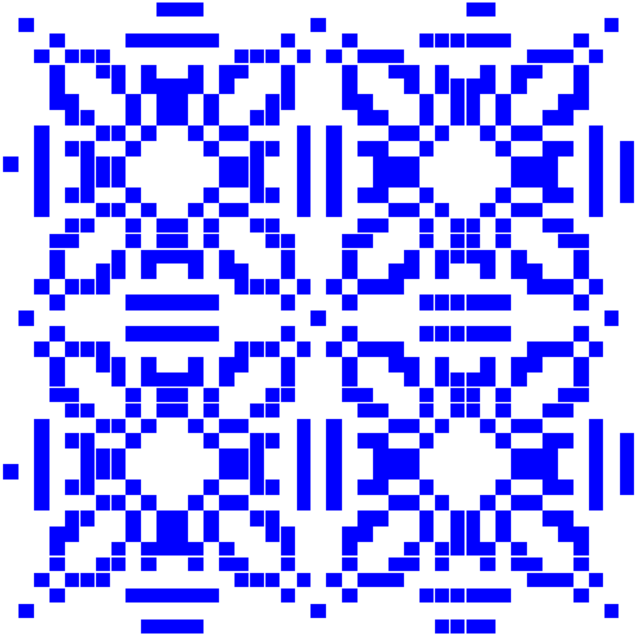
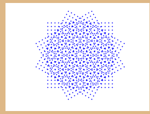


Figure 6: Federico's Same Example, Recreated With His Methods.



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