Fractal Tilings

Katie Moe and Andrea Brown

December 13, 2006

Table of Contents

Introduction

In this presentation we will be generating tilings with individual tiles called *fractiles* whose boundaries are fractal curves. Fractal curves are objects or quantities that display self-similarity, in a somewhat technical sense, on all scales. This means that it looks the same at any scale. We will use an iterative process, involving repeated compositions of two or more functions and those, in turn, will generate the fractal tiling.

Examples of Fractal Tilings

- Start with a matrix $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are chosen so that $a^2 + b^2 > 1$.
- We must understand that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are points in the complex plane and $M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 - bx_2 \\ ax_1 + bx_2 \end{bmatrix}$ represents the complex multiplication of $x_1 + ix_2$ by $a + ib$.

- Next, we must find a collection of vectors that will translate the copies of the *fractile* so that they are positioned correctly in the tiling.
- We will define the set $\xi = \{\mathbf{r}_j\}$ and the vectors in this set have integer coordinates that lie in or on S but not on the two outer edges that don't have the origin as a vertex. ξ has exactly m vectors.
- The unit square that is determined by the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is mapped onto the square S with area $m = a^2 + b^2$ and is spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$.

Example 1

- Let $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ then $m = 2$.
- We can determine that the two translation vectors are $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

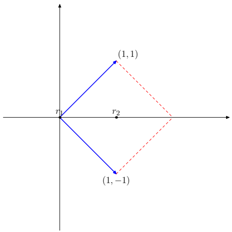


Figure: Finding Equivalent Residue Vectors.

- Now we have $\xi = \{\mathbf{r}_1, \mathbf{r}_2\}$.
- For $z = (x_1, x_2)$, where z is our initial point of translation, we can define our mappings as $f_j(\mathbf{z}) := \mathbf{r}_j + M^{-1}(\mathbf{z})$ for $j = 1, 2$. That is,

$$f_1 := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} .5 & -.5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f_2 := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} .5 & -.5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The collections of functions $\{f_j\}$ is called an iterated function system. To initiate this process an initial point z_o is randomly selected in the plane and is used to evaluate $f_1(z_o)$ and $f_2(z_o)$. For $n \geq 1$, we make sure to choose recursively and randomly so that $z_n \epsilon \{f_1(z_{n-1}), f_2(z_{n-1})\}$.

Points will be lying near the tiling after a few iterations, but thousands of iterations will be needed to generate the desired tiling. The result of the iterated function system for this example can be seen in the following Figure.

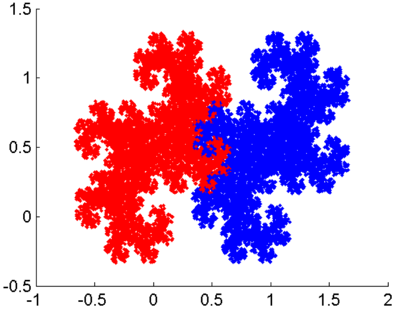


Figure: Residue Vectors.

Example 2

If we have $M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

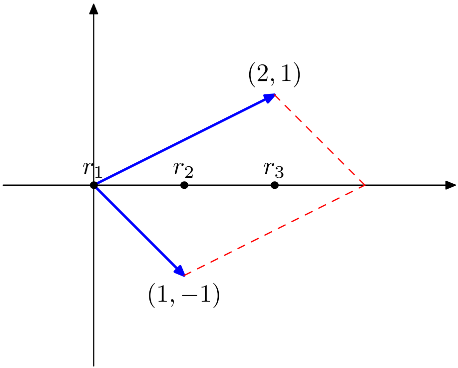


Figure: Residue Vectors.

The tiling produced will be three tiles stacked horizontally.

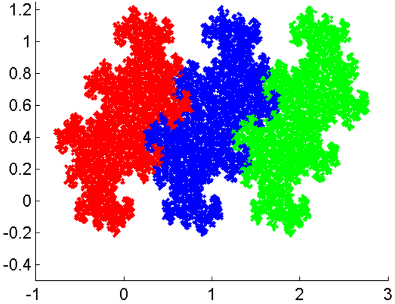


Figure: Horizontal Tiling.

Creating the Tilings

- To generate a tiling we need a matrix to be an invertible integer matrix that is an expansive map, i.e. all eigenvalues have modulus larger than 1.
- The matrix we will choose will be $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- The translation vectors are chosen with the following process. For a matrix M as above, $|det(M)| = |ad - bc| = m$ is the area of parallelogram P spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$.
- These vectors are called principal residue vectors. The vectors in $\{\mathbf{r}_j\}$ form a complete residue system for M.

- Generally, as long as $\mathbf{y}_1 = \mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_j \approx \mathbf{r}_j$ for $j = 2, ...m$, then the collection of vectors $\{\mathbf{y}_j\}$ will also form a complete residue system for matrix M.
- The location of the residue vectors determines the locations of the fractiles but the shape of the tilings may change drastically with the different choices of residue systems.

Short Summary of Important Ideas

- M represents an expansive map
- $\{\mathbf{y}_1, ...\mathbf{y}_m\}$ is a complete residue system for M
- $f_j(\mathbf{z}) := \mathbf{r}_j + M^{-1}(\mathbf{z})$.
- The attractor set $A = \cup_m^{j=1} A_j$ is the tiling of m tiles A_j. These tiles are now called m-rep tiles.

These ideas will now be used to create a tiling of m-rep tiles.

Example 3

Let $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$; then $m = 5$. Here the principal residue vectors are $r_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $r_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $r_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $r_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

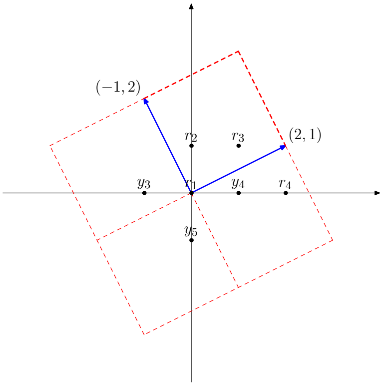


Figure: Residue Vectors.

Example 3

For a more symmetric tiling, we choose the following equivalent residue vectors for our residue system out of the collection $\{\mathbf{y}_j\}$. Our next tiling is created by using $\mathbf{y}_1 = \mathbf{r}_1, \mathbf{y}_2 = \mathbf{r}_2,\ \mathbf{y}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \approx \mathbf{r}_3,\ \mathbf{y}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \mathbf{r}_4,$ and $\mathbf{y}_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \approx \mathbf{r}_5$. The vectors $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5\}$ are symmetric about $\mathbf{r}_1$.

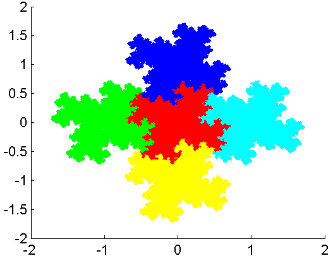


Figure: Residue Vectors.

Tiles with Radial Symmetry

When $m = 2, 3, 4, 5,$ and 7, we are able to create a tiling that has radial symmetry.
In order to have radial symmetry we need a change of base matrix (B).

Example 4

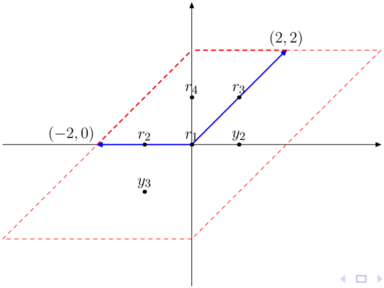
Let $M = \begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$. New residue vectors

$$B\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad B\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad B\mathbf{y}_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad B\mathbf{y}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are formed by the equation

$$f_j(\mathbf{z}) = B\mathbf{y}_j + h^{-1}(\mathbf{z})$$

where $h = BMB^{-1}$.



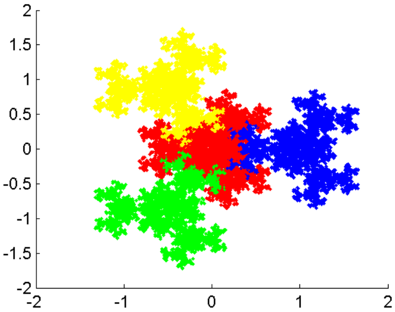
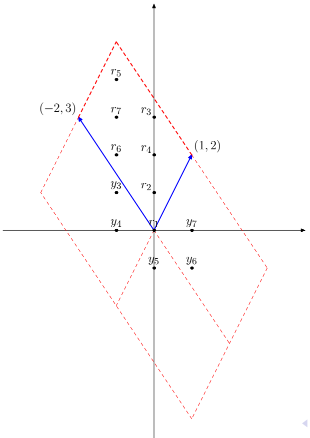


Figure: Horizontal Tiling.

Example 5

$$M = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1/2 \\ 0 & -\sqrt{3}/2 \end{bmatrix}$$

$$B\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; B\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; B\mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; B\mathbf{y}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \; B\mathbf{y}_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \; B\mathbf{y}_6 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; B\mathbf{y}_7 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



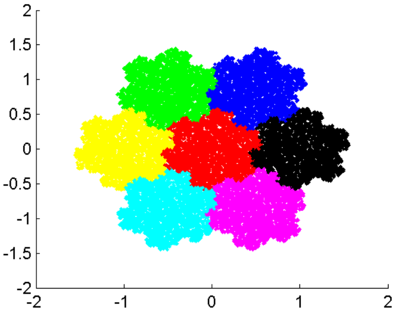


Figure: Residue Vectors.

Similarity Maps

There are two cases when you are developing similarity maps:

- M has two real eigenvalues with independent eigenvectors
- M has a pair of complex conjugate eigenvalues

The format

$$f_j(\mathbf{z}) = B\mathbf{y}_j + h^{-1}(\mathbf{z})$$

where $h = BMB^{-1}$ and B^{-1} is the eigenvectors is used.

Example 6

$$M = \begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1/2 \\ 0 & -\sqrt{2}/2 \end{bmatrix}$$

$$B\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; B\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; B\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \; B\mathbf{y}_4 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \; B\mathbf{y}_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; B\mathbf{y}_6 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

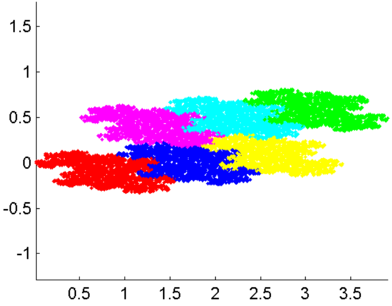


Figure: Similarity Tiling.

Example 7

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 1 \\ (\sqrt{6}-2)/2 & -(\sqrt{6}+2)/2 \end{bmatrix}$$

$$B\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; B\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; B\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

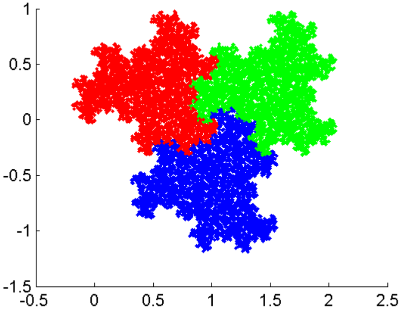
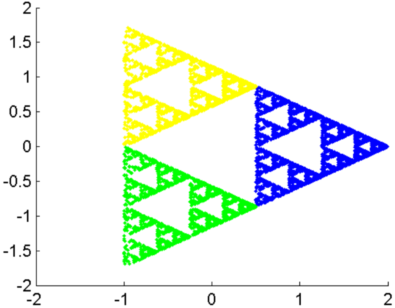
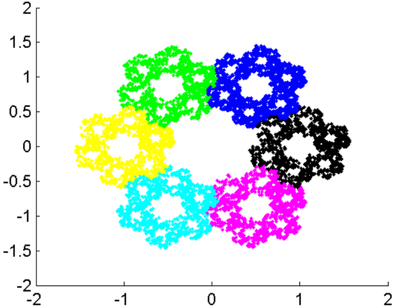


Figure: Similarity Tiling.



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1
0
1
2


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1
0
1
2

Fractal are fun! (and pretty)