

# Modeling Prey-Predator Populations

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# Introduction

In the biological world species interact with each other. When the populations of prey and predator species interact, they each influence the population of the other. Predators lower the prey population, and prey help the predator population grow.

In our presentation, the populations we will be discussing are uninfluenced by any other populations. This unlikely hypothetical situation may be represented by a population of aardvarks and ants living on a stranded island.

We will explore this situation for the purpose of understanding the general influence prey and predator populations have on each other in the real world.

## Our Populations and Equations: Our Populations

The prey in this model will be ants, and the predators armadillos. Let  $P$  represent the ants and  $Q$  represent the armadillos. Let  $P_t$  represent the size of the prey population and  $Q_t$  represent the size of the predator population at time  $t$ .

The equations used to show change in population are represented as

$$\Delta P = F(P, Q)$$

And

$$\Delta Q = G(P, Q)$$

# Our Equations

The population of the Ants without the influence of Aardvarks is represented by

$$\Delta P = rP(1 - P/K)$$

Where

$$r, K$$

are positive constants. And the of Aardvarks without the presence of ants is represented by

$$\Delta Q = -uQ$$

Where

$$0 < u < 1$$

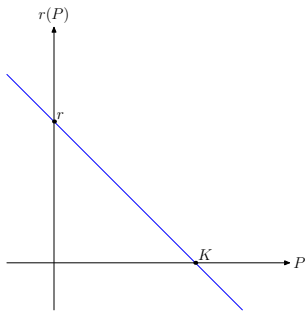


Figure: The rate of growth of  $P$  as a function of  $P$ .

This graph shows that the prey population would not grow infinitely but there is a finite number of where they population would no longer grow but stay the same. This is where our equation for the prey comes from.

Modeling the the two populations with the amount of interaction between the ants and aardvarks the equations change. The ants are represent by the equation:

$$\Delta P = rP(1 - P/K) - sPQ$$

and the aardvarks are represented by the equation:

$$\Delta Q = -uQ + vPQ$$

where  $v$  and  $s$  are positive constants.

We now look at the equations differently, so that  $Q$  becomes  $Q_t$ ,  $P$  becomes  $P_t$ ,  $\Delta Q = Q_{t+1} - Q_t$  and  $\Delta P = P_{t+1} - P_t$

$$\Delta P = rP(1 - P/K) - sPQ$$

$$P_{t+1} - P_t = rP_t(1 - P_t/K) - sP_tQ_t$$

$$P_{t+1} = P_t(1 + r(1 - P_t/K)) - sP_tQ_t$$

and  $\Delta Q$  can be expressed the same way.

$$Q_{t+1} = (1 - u)Q_t + P_tQ_t$$

Remember  $r, s, u, v$ , and  $K$  are positive constants and  $u \leq 1$

Introducing numbers into our equations. Let

$$K = 1, r = 1.3, s = .5, u = .7, \text{ and } v = 1.6$$

Our equations are now:

$$P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t$$

$$Q_{t+1} = .3Q_t + 1.6P_tQ_t$$

# Eigenvalues and Model Behavior: Eigenvalues

**Eigenvalues**— let's stop the idea of population modeling and take a look at eigenvalues. We are looking at the meaning of eigenvalues in linear system of two equation and two unknowns and how they effect the behavior of their graphs.

Let our linear system be represented in terms of the vector  $\mathbf{x}_n$ , which we manipulate with the matrix  $A$  which gives us the vector  $\mathbf{x}_{n+1}$  with our next values, which are represented by:

$$\mathbf{x}_{n+1} = A\mathbf{x}_n$$

There are five ways for the convergence of our system There are sinks, sources, spiral sinks,spiral sources, and saddle points. Assuming that:

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$

and

$$A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$

and  $\lambda_1 \neq \lambda_2 \Rightarrow \mathbf{v}_1$  and  $\mathbf{v}_2$  are independent. This means that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  span  $\mathbb{R}^2$  and  $\mathbf{x}_0 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ .

A little math that we will skim over shows:

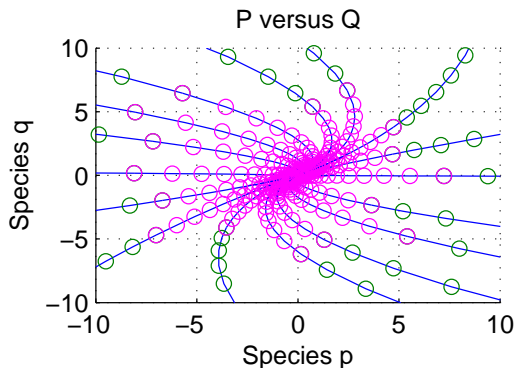
$$\begin{aligned}\mathbf{x}_1 &= A\mathbf{x}_0 \\ &= A(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) \\ &= \alpha_1(A\mathbf{v}_1) + \alpha_2(A\mathbf{v}_2) \\ &= \alpha_1\lambda_1\mathbf{v}_1 + \alpha_2\lambda_2\mathbf{v}_2\end{aligned}$$

We move on to  $\mathbf{x}_2$ , which will show the beginnings of a pattern:

$$\begin{aligned}\mathbf{x}_2 &= A\mathbf{x}_1 \\ &= A(\alpha_1\lambda_1\mathbf{v}_1 + \alpha_2\lambda_2\mathbf{v}_2) \\ &= \alpha_1\lambda_1A\mathbf{v}_1 + \alpha_2\lambda_2A\mathbf{v}_2 \\ &= \alpha_1\lambda_1(A\mathbf{v}_1) + \alpha_2\lambda_2(A\mathbf{v}_2) \\ &= \alpha_1\lambda_1^2\mathbf{v}_1 + \alpha_2\lambda_2^2\mathbf{v}_2\end{aligned}$$

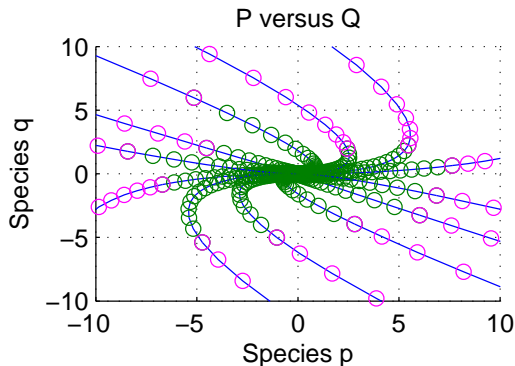
With repetition we will eventually discover that  $\mathbf{x}_n$  is, in fact following the  $\lambda^n$  pattern that we can see emerging.

## Model Behavior: Source



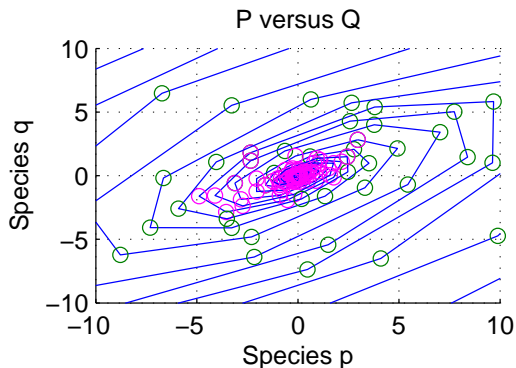
The pink spots show the pattern going back in time, the pink in the middle shows that this is a source, showing that the population is unstable because it diverges to infinity. The source is made by  $\lambda_1, \lambda_2 > 1$

# Sink



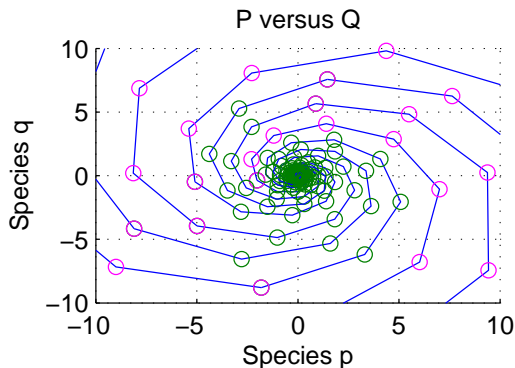
The green center shows that this is a sink. This shows a stable population because it goes to zero. The sink is made with  $\lambda_1, \lambda_2 < 1$

# Spiral Source



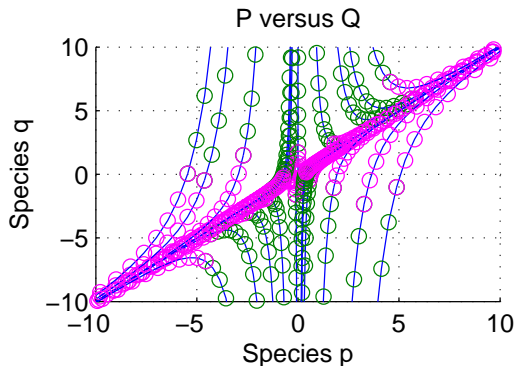
This shows a spiral source. This occurs when the eigenvalues are complex numbers that have a magnitude greater than one.  $\lambda_1 = a + bi$ , and  $\lambda_2 = a - bi$ , where  $\|\lambda_1\| = \|\lambda_2\| > 1$

# Spiral Sink



This shows a spiral sink. This occurs when the eigenvalues are complex numbers that have a magnitude less than one.  $\lambda_1 = a + bi$ , and  $\lambda_2 = a - bi$ , where  $\|\lambda_1\| = \|\lambda_2\| < 1$

# Saddle Point



This shows a saddle point which occurs when one eigenvalue is greater than 1 and the other is less than 1.  $\lambda_1 < 1$  and  $\lambda_2 > 1$

# Equilibria and Eigenvalues

Let's return to our population modeling. When looking at the growth and decrease of the population, we wonder whether they will keep changing or if they will reach a stable point and stay there. The stable is what we call the equilibrium point.

# Equilibria

An **Equilibrium Point** is where  $P_{t+1} = P_t$  and  $Q_{t+1} = Q_t$ ; each new expression is then a repeat of the previous expression. We now look at our equations from before but we replace  $P_{t+1}$  and  $P_t$  with  $P^*$  and let  $Q^*$  represent  $Q_{t+1}$  and  $Q_t$  our equations are now changed to:

$$P^* = P^*(1 + r(1 - P^*/K)) - sP^*Q^*$$

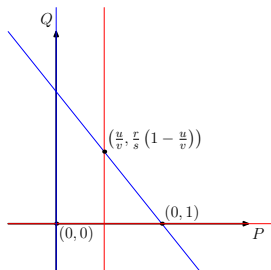
$$Q^* = (1 - u)Q^* + vP^*Q^*$$

We are making the carrying capacity  $K = 1$ .

Now we take our equations and we simplify our equations change to:

$$0 = P^*(r(1 - P^*) - sQ^*)$$

$$0 = Q^*(-u + vP^*)$$



This means that  $Q^* = 0$ ,  $P^* = 0$ ,  $P^* = \frac{u}{v}$ , and/or  $Q^* = \frac{r}{s}(1 - P)$ . With these conditions we can create what are called nullclines. These nullclines will be the general image of what these conditions look like on a set axes that we call  $P$  and  $Q$ .

The first two equilibria,  $(0, 0)$  and  $(1, 0)$  are points that we do not care about. In both interactions at least one population is extinct. Instead we focus on a third point:  $(\frac{u}{v}, \frac{r}{s}(1 - \frac{u}{v}))$ . This will change depending on our values for  $u, v, r,$  and  $s$ . If we let  $u = .7, v = 1.6, r = 1.3,$  and  $s = .5$  Then

$$P^* = u/v = .4375$$

and

$$Q^* = r/s(1 - u/v) = 1.4625$$

Once we have our equilibrium point we can linearize the equations for prey and predator populations. We represent the equations by

$$P_t = P^* + p_t$$

and

$$Q_t = Q^* + q_t$$

where  $p_t$  and  $q_t$  represent the distance of the point to the equilibrium point. We are now considering the model very close to the equilibrium point and the values of  $p_t$  and  $q_t$  very small.

We now have two equations for our model very near the equilibrium point:

$$p_{t+1} = .43125p_t - .21875q_t - 1.3p_t^2 - .5p_tq_t$$

$$q_{t+1} = 2.34p_t + q_t + 1.6p_tq_t$$

We have gone from one nonlinear system to another nonlinear system. But because  $p_t$  and  $q_t$  are really really really small, we can throw away  $p_tq_t$  and  $p_t^2$  as negligible.

Our system really(almost) looks like this:

$$p_{t+1} = .43125p_t - .21875q_t$$

$$q_{t+1} = 2.34p_t + q_t$$

We finally have a linear system which we can use to create the coefficient matrix  $A$ , so that  $A$  equals

$$\begin{bmatrix} .43125 & -.21875 \\ 2.34 & 1 \end{bmatrix}$$

The eigenvalues of  $A$  are determined by the characteristic polynomial:

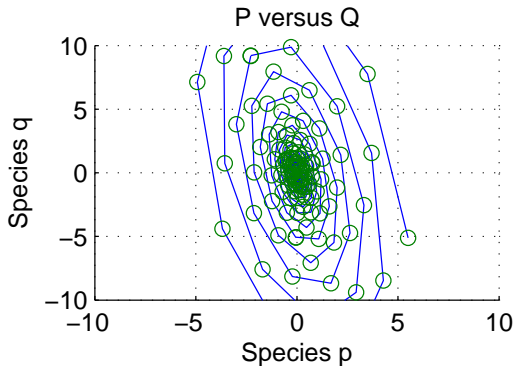
$$0 = \lambda^2 - T(A)\lambda + D(A)$$

With the trace and the determinant of the characteristic polynomial, we have a simple quadratic in terms  $\lambda$

$$0 = \lambda^2 - 1.43125\lambda + .943125$$

Solving for  $\lambda$  we get

$$\lambda \approx .71563 \pm .65651i$$



The magnitude of these two eigenvalues is  $\sqrt{.71563^2 + .65651^2}$ . This is approximately .97115. Because the magnitudes of the two eigenvalues are less than one and they are complex numbers the model behaves as a spiral sink. The populations are stable around our equilibrium point. Which is represented in the graph above. The origin in this figure is the equilibrium point (.4375,1.4625).