Image Metamorphosis by Affine Transformations

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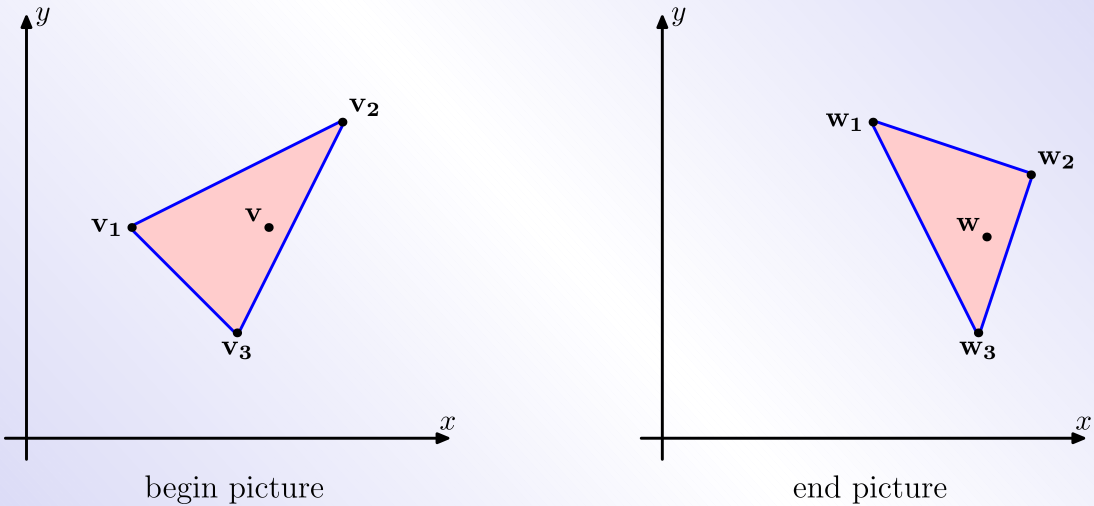
Objectives

- Triangulation
- Linear Transformations
- Affine Transformations
- Warps
- Time-Varying Warps
- Picture Density
- Morphs

Triangulation

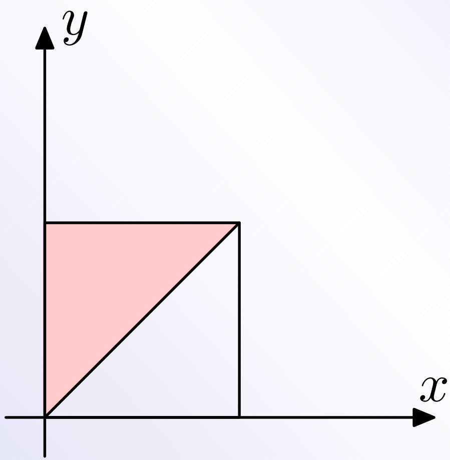


We will map each enclosing triangle from the *begin picture* to the *end picture*.

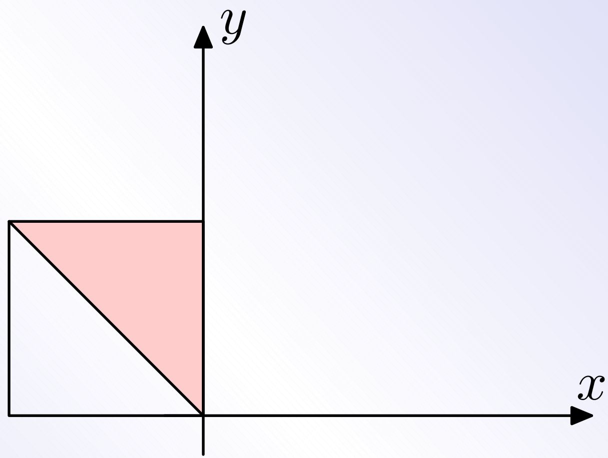
Linear Transformations

Standard Matrices

By multiplying a vector in $\mathbb{R}^2$ with a 2-by-2 matrix, called a standard matrix, we can reflect, rotate, compress, expand, or shear the shape of the unit square.



Reflections



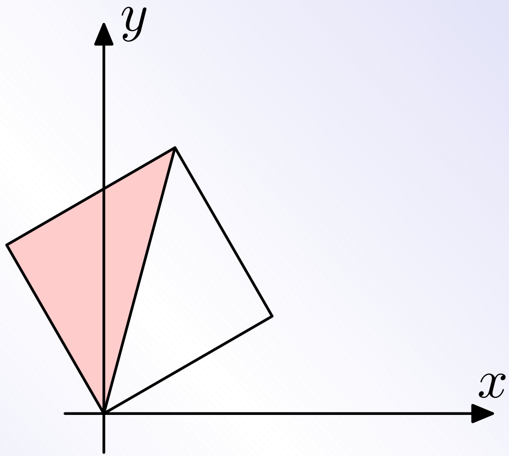
Note that the standard matrix multiplies every vector in the square, therefore it changes the shape of the square. Reflections and rotations are effects a standard matrix can have on a vector. The matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

creates a reflection about the y-axis.

Back
Close

Rotations

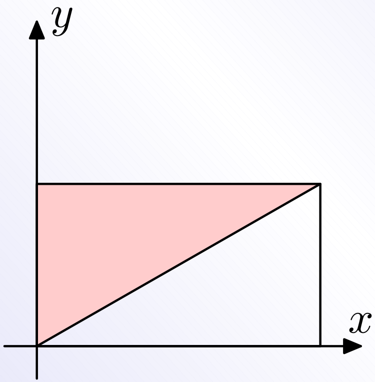


The matrix

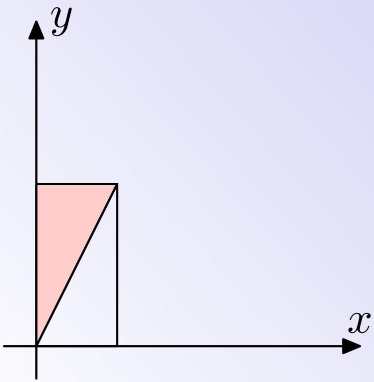
$$A = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

creates a rotation of θ about the origin.

Expansions and Compressions



Expansion

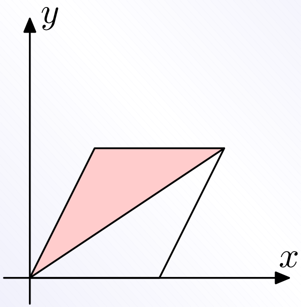


Compression

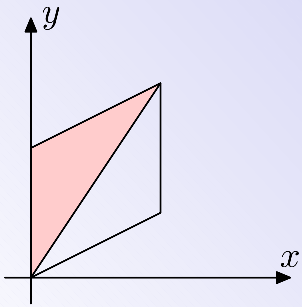
If the x-coordinate of a vector is multiplied by k an expansion or a compression is obtained in the x-direction, where $0 < k < 1$ provides a compression and if $k > 1$ an expansion is obtained. If the y-coordinate is multiplied, the y-direction is manipulated. The standard matrix is

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

Shears



Shear in x-direction

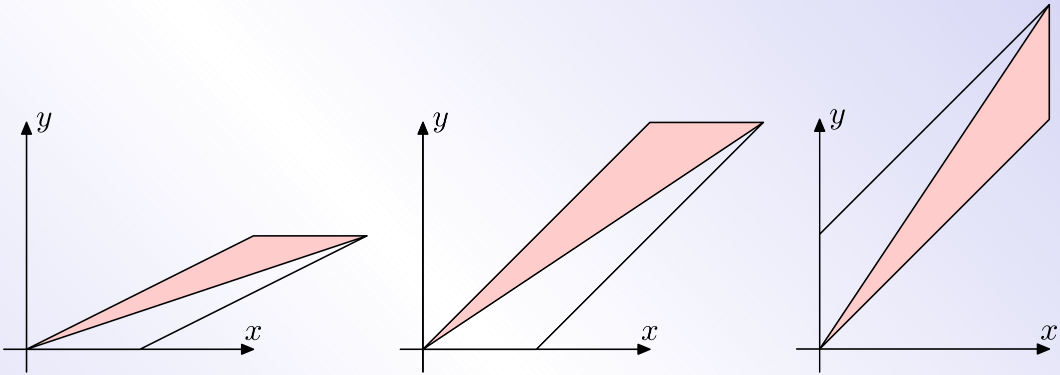


Shear in y-direction

A shear in the x-direction is defined as a transformation that moves each point (x, y) parallel to the x-axis by an amount ky to the new position $(x + ky, y)$. Similarly, a shear in the y-direction moves each point (x, y) parallel to the y-axis by an amount kx to the new position $(x, y + kx)$. The standard matrices for these operations are

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Combining Basic Transformations



Shear in x-direction　　Expansion in y-direction　　Reflection

To obtain more complex transformations these different types of basic transformations can be combined. In this example we combine a shear in the x-direction, an expansion in the y-direction, and a reflection about the line $y = x$.

Affine Transformations

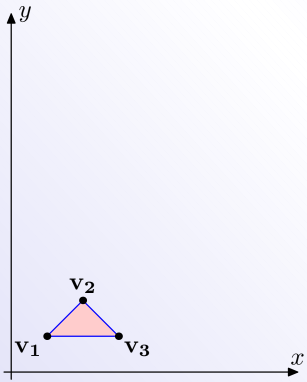
In order to get all the indices of our end triangle we can multiply each point of the begin-triangle with a specific 2-by-2 standard matrix and add a two-dimensional vector to the result that moves the entire triangle a certain distance. The new indices of the end-triangle are given by

$$\mathbf{w} = M\mathbf{v} + \mathbf{b}$$

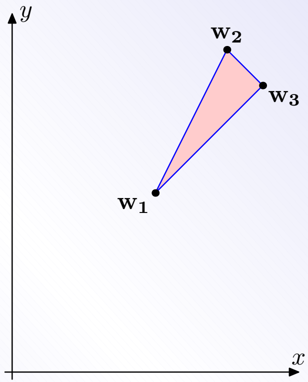
where $\mathbf{v}$ represents any coordinate point in the begin picture and M is the standard matrix and $\mathbf{b}$ is the two-dimensional vector. M and $\mathbf{b}$ can be found by a system of equation from the vertex points of the begin- and end-triangle

An Example

Lets do an example and see what the standard matrix does and what the two-dimensional vector does. The goal is to change a triangle into another shape and position.

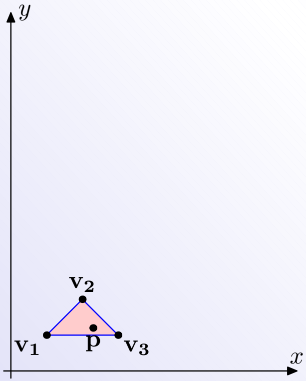


Begin Triangle

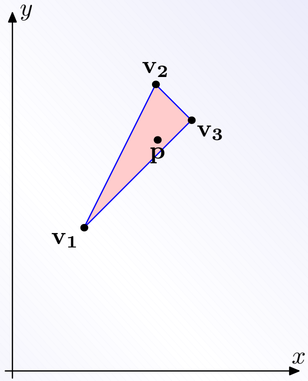


End Triangle

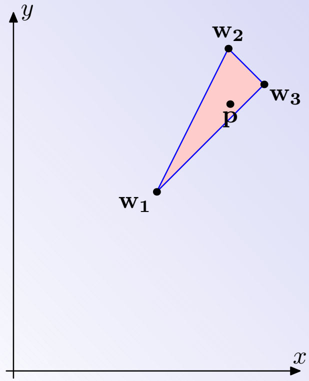
An Example



This is the triangle we are starting with



Triangle multiplied by M



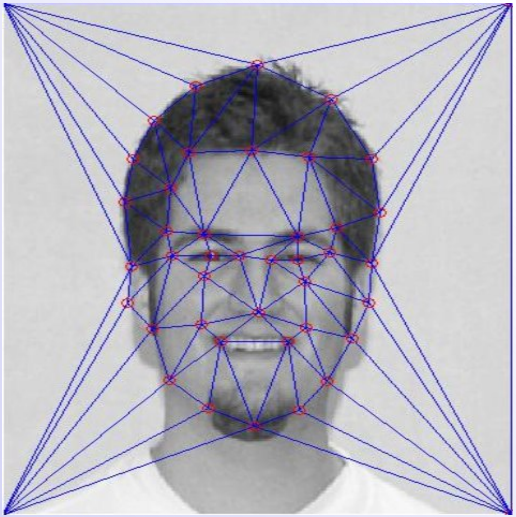
Finally, the vector $\mathbf{b}$ is added

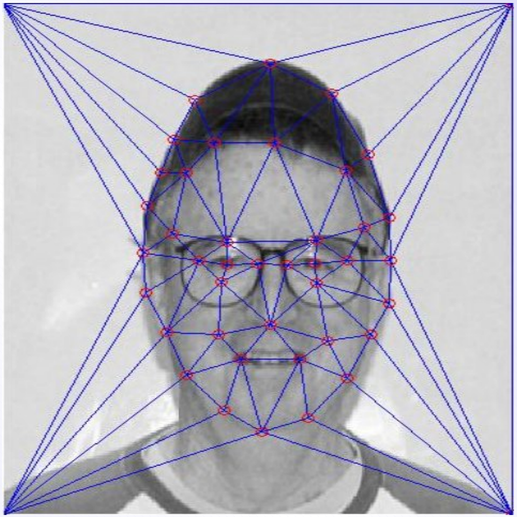
By setting up a system of equation we found the matrix M and the vector $\mathbf{b}$. The first picture is obtained by multiplying the triangle by M. As you can see the shape is good, but it's in the wrong place. To fix that we have to move the triangle by $\mathbf{b}$ and we end up with the third picture. We successfully warped the triangle.

Back
Close

Triangulation of an Image

What we did to one triangle we are going to apply to lots of triangles, that each have a segment of a picture inside of them. Imagine the two pictures consisting of 82 triangles. There are 82 affine transformations that have to be done to warp the entire picture.





Warp of the Image

As a result the begin-picture changed into the shape of the end-picture.

Time-Varying Warps

Let $\mathbf{v}_i$ and $\mathbf{w}_i$ be the i-th vertex point of a triangle. We can move the vertex points along a line from $\mathbf{v}_i$ to $\mathbf{w}_i$. As time elapses from $t = 0$ to $t = 1$ we can define the i-th vertex point that moves at a constant velocity from the start position to the end position as

$$\mathbf{u}_i(t) = (1 - t)\mathbf{v}_i + t(\mathbf{w}_i). \tag{1}$$

So $\mathbf{u}_i(0) = \mathbf{v}_i$ and $\mathbf{u}_i(1) = \mathbf{w}_i$ and for any value t in between 0 and 1 we can apply Equation (1) and find all the vertex points at any given time t.

With our new set of intermediate vertex points we can do the affine transformation from the begin-picture to the intermediate picture and when we incrementally change time t from 0 to 1 we obtain a set of frames with can be made into a motion picture.

Time-Varying Warps

If we do a time-varying warp with the begin-picture as well as with the end-picture and then reverse the order of the sequence, we have the basis for a morph because we have a set of pictures that have the same shape. What we need to do now is blend these pairs together in a matter where the gray scale values change from the ones of the end-picture to the ones in the end picture.

Time-Varying Warps

$t = 0$ | $t = 0.25$ | $t = 0.50$ | $t = 0.75$ | $t = 1$

Back
Close

Picture Density

The concentration of color at any given point of an image is referred to as the picture density ρ. In our gray scale images the picture density ranges from 0 for black to 255 for white. For our morph we want all of the points to be smoothly changing ρ from ρ_0 which is the picture density in the begin-picture to ρ_1 which is the picture density in the end-picture as time . The equation to obtain ρ at each frame is

$$\rho(u) = (1 - t)\rho_0 + t\rho_1$$

where t is the time that has passed and u is the position of a pixel at time t as it traverses from it's starting location to it's destination. So for example at time $t = 0.25$ ρ is equal to 0.75 of the picture density from the start and 0.25 of the picture density from the end.

Morphs

Finally we reach our goal by using our pair of time-varying warps and blending each pair of pictures together by using the formula for the picture density.

$t = 0$

$t = 0.25$

$t = 0.50$

$t = 0.75$

$t = 1$