Fourier Series and the Discrete Fourier Expansion

Matthew Lincoln

Adrienne Carter

sillyajc@yahoo.com

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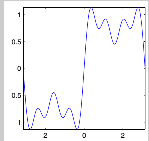
Abstract

This article is intended to introduce the Fourier series and the Discrete Fourier Expansion and demonstrate their power through examples.

1. Introduction

Like the function in Figure 1, any function can be approximated by an infinite series of sine and cosine functions. As we can see from the Figure 2, the green, red, cyan, and violet are the basis for the blue function. A different view is provided in Figure 3.



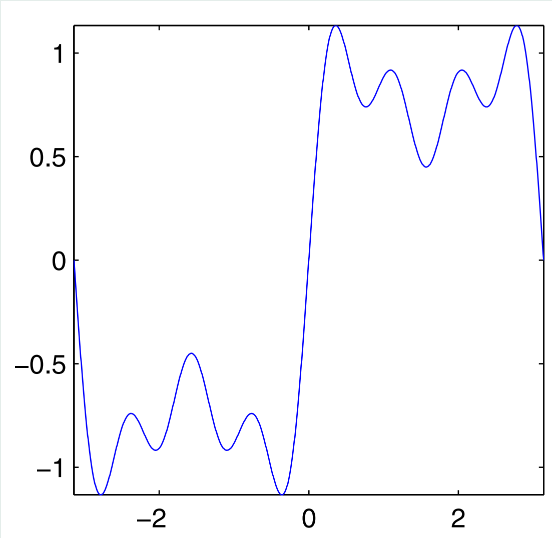
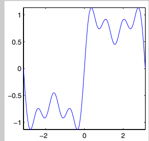


Figure 1: This complex wave is a combination of five independent sinusoidal functions varying in amplitude and frequency.



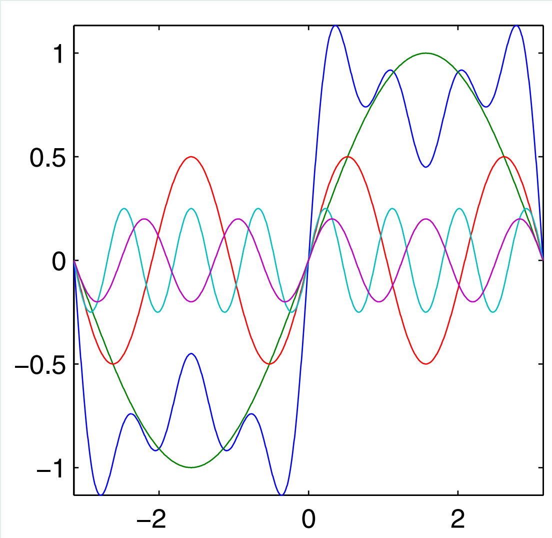
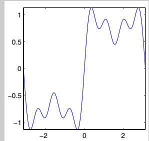


Figure 2: The blue wave is the sum of the four simple harmonic functions.



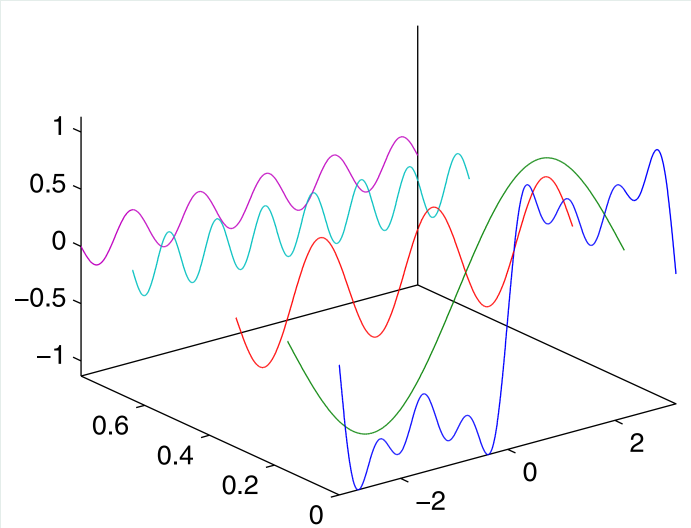
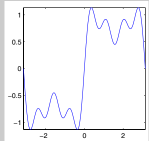


Figure 3: A three dimensional view of the functions summed and the resulting “complex” wave.



1.1. The Inner Product

Before we begin our discussion of a basis, we need to broaden our definition of the inner product. The inner product, or dot product, of two vectors results in a scalar. Similarly, the dot product of two functions, given the limits, will also yield a number. The inner product of two functions is defined as the integral of the product of those functions. In symbols,

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx.$$

For this to be an inner product, it must satisfy these five restraints that define the dot product:

1. $\langle f, f\rangle \geq 0;$
2. $\langle f, f\rangle = 0, \text{ iff } f = 0;$
3. $\langle f, g\rangle = \langle g, f\rangle;$
4. $\langle \alpha f, g\rangle = \langle f, \alpha g\rangle = \alpha\langle f, g\rangle;$
5. $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle.$

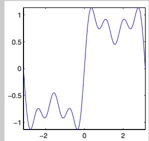
Each of these properties can be easily proven.

1.

$$\begin{aligned}\langle f, f\rangle &= \int_a^b [f(x)]^2 dx \\ &\geq 0\end{aligned}$$

2.

$$\begin{aligned}\langle f, f\rangle &= \int_a^b [f(x)]^2 dx \\ &= 0, \text{ iff } f(x) = 0\end{aligned}$$



3.

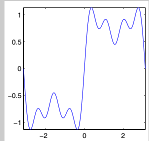
$$\begin{aligned}
\langle f, g\rangle &= \int_a^b f(x)g(x)\,dx \\
&= \int_a^b g(x)f(x)\,dx \\
&= \langle g, f\rangle
\end{aligned}$$

4.

$$\begin{aligned}
\langle \alpha f, g\rangle &= \int_a^b \alpha f(x)g(x)\,dx \\
&= \int_a^b f, \alpha g dx \\
&= \langle f, \alpha g\rangle \\
&= \int_a^b \alpha\langle f, g\rangle dx \\
&= \alpha\langle f, g\rangle
\end{aligned}$$

5.

$$\begin{aligned}
\langle f, g+h\rangle &= \int_a^b f(x)\left(g(x)+h(x)\right)\,dx \\
&= \int_a^b f(x)g(x)+f(x)h(x)\,dx \\
&= \int_a^b f(x)g(x)dx + \int_a^b f(x)h(x)\,dx \\
&= \langle f, g\rangle + \langle f, h\rangle
\end{aligned}$$



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1.2. Orthogonality of the Basis

We will use

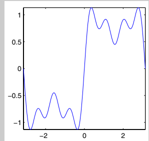
$$\{1, \cos(1x), \sin(1x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots\}, \quad n = 0, 1, 2, \ldots$$

as a basis. Our first task is to show that distinct basis elements are orthogonal. That is, we must show that

$$\begin{aligned} \langle 1, \cos nx \rangle &= 0, \\ \langle 1, \sin nx \rangle &= 0, \\ \langle \cos mx, \cos nx \rangle &= 0, \;\; m \neq n; \\ \langle \sin mx, \sin nx \rangle &= 0, \;\; m \neq n; \\ \langle \cos mx, \sin nx \rangle &= 0. \end{aligned}$$

Note n and m are integers. It is important that when we dot a cosine function with another cosine function, or sine with sine, we have different frequencies. These different frequencies must be integer multiples of the original frequency. The inner product of a cosine function and a sine function has no restriction on the frequency.

We perform all the integrations over the period, represented here from $[0, 2\pi]$.



1. If $n\epsilon\{1,2,3,\ldots,\}$, then

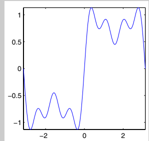
$$\begin{aligned}\langle 1, \cos nx\rangle &= \int_0^{2\pi} \cos nx\,dx \\ &= \int_0^{2\pi} \cos nx\,dx \\ &= \frac{1}{n}\sin nx\Big|_0^{2\pi} \\ &= \frac{1}{n}\left[\sin(2n\pi) - \sin 0\right] \\ &= \frac{1}{n}(0-0) \\ &= 0\end{aligned}$$

Therefore, 1 and $\cos nx$ are orthogonal.

2. If $n\epsilon\{1,2,3,\ldots,\}$, then

$$\begin{aligned}\langle 1, \sin nx\rangle &= \int_0^{2\pi} \sin nx\,dx \\ &= -\frac{1}{n}\cos nx\Big|_0^2\pi \\ &= -\frac{1}{n}\left[\cos(2n\pi) - \cos 0\right] \\ &= -\frac{1}{n}(1-1) \\ &= -\frac{1}{n}(0) \\ &= 0\end{aligned}$$

Therefore, 1 is orthogonal to $\sin nx$.



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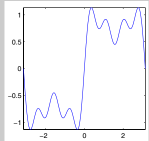
3. If $n \neq m$, then

$$
\begin{aligned}
\langle \cos mx, \cos nx \rangle &= \int_0^{2\pi} \cos mx \cos nx \, dx \\
&= \frac{\sin((n-m)x)}{2(n-m)} + \frac{\sin((n+m)x)}{2(n+m)} \bigg|_0^{2\pi} \\
&= \left[\frac{\sin(2\pi(n-m))}{2(n-m)} + \frac{\sin 2\pi(n+m)}{2(n+m)} \right] \\
&\quad - \left[\frac{\sin(0(n-m))}{2(n-m)} + \frac{\sin(0(n+m))}{2(n+m)} \right] \\
&= (0+0) - (0+0) \\
&= 0
\end{aligned}
$$

Therefore, $\cos mx$ and $\cos nx$ are orthogonal, provided $m \neq n$.

4. If $m \neq n$, then

$$
\begin{aligned}
\langle \sin mx, \sin nx \rangle &= \int_a^{2\pi} \sin mx \sin nx \, dx \\
&= -\frac{\sin((n-m)x)}{2(n-m)} + \frac{\sin(n+m)x}{2(n+m)} \bigg|_0^{2\pi} \\
&= \left[-\frac{\sin(2\pi(n-m))}{2(n-m)} + \frac{\sin(2\pi(n+m))}{2(n+m)} \right] \\
&\quad - \left[-\frac{\sin(0(n-m))}{2(n-m)} + \frac{\sin(0(n+m))}{2(n+m)} \right] \\
&= (0+0) - (0+0) \\
&= 0
\end{aligned}
$$



Therefore, $\sin mx$ and $\sin nx$ are orthogonal if $m \neq n$.

5. If n, m are any positive integers, then

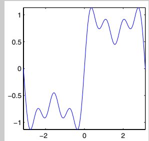
$$\begin{aligned}
\langle \cos mx, \sin nx \rangle &= \int_0^{2\pi} \cos mx \sin nx \, dx \\
&= -\frac{\cos((n-m)x)}{2(n-m)} - \frac{\cos((n+m)x)}{2(n+m)} \bigg|_0^{2\pi} \\
&= \left[-\frac{\cos(2\pi(n-m))}{2(n-m)} + \frac{\cos(2\pi(n+m))}{2(n+m)} \right] \\
&\quad - \left[-\frac{\cos(0(n-m))}{2(n-m)} + \frac{\cos(0(n+m))}{2(n+m)} \right] \\
&= \left(-\frac{1}{2(n-m)} + \frac{1}{2(n+m)} \right) - \left(-\frac{1}{2(n-m)} + \frac{1}{2(n+m)} \right) \\
&= 0
\end{aligned}$$

Therefore, $\cos mx$ and $\sin nx$ are orthogonal at any frequency.

Thus, distinct members of the basis are orthogonal. Now, what happens when you take the inner product of a basis element with itself?

1. First, compute $\langle 1, 1 \rangle$.

$$\begin{aligned}
\langle 1, 1 \rangle &= \int_0^{2\pi} 1 \, dx \\
&= [x|_0^{2\pi} \\
&= 2\pi - 0 \\
&= 2\pi
\end{aligned}$$



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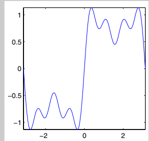
2. Secondly, compute $\langle \cos nx, \cos nx \rangle$.

$$
\begin{aligned}
\langle \cos nx, \cos nx \rangle &= \int_0^{2\pi} \cos^2 nx \, dx \\
&= \int_0^{2\pi} \left(\frac{1 + \cos 2nx}{2} \right) dx \\
&= \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2nx}{2} \right) dx \\
&= \left[\frac{1}{2} x + \frac{\sin 2nx}{4n} \right|_0^{2\pi} \\
&= \frac{1}{2} 2\pi + \left[\frac{\sin 2n(2\pi)}{4n} - \frac{\sin 2n(0)}{4n} \right] \\
&= \frac{2\pi}{2} + (0 - 0) \\
&= \pi
\end{aligned}
$$



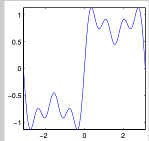
3. Finally, compute $\langle \sin nx, \sin nx \rangle$.

$$\begin{aligned}
\langle \sin nx, \sin nx \rangle &= \int_0^{2\pi} \sin^2 nx\, dx \\
&= \int_0^{2\pi} \frac{1 - \cos 2nx}{2}\, dx \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{\cos 2nx}{2} \right) \\
&= \left[\frac{1}{2}x - \frac{\sin 2n}{4n} \right|_0^{2\pi} \\
&= \frac{1}{2}(2\pi) - \left[\frac{\sin 2n(2\pi)}{4n} - \frac{\sin 2n(0)}{4n} \right] \\
&= \frac{2\pi}{2} - (0 - 0) \\
&= \pi
\end{aligned}$$

2. The Fourier Series

Any function can be written as an infinite sum of sine and cosine functions. Jean Baptist Fourier was the first mathematician to write this as an expression. It is now called the Fourier series and is defined as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right),$$



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where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx,$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \text{ and,}$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx.$$

The integration is performed over the period, stated in the individual terms as $[0, 2\pi]$. It can just as easily be represented as $[-\pi, \pi]$. We will use both expressions of the period in the later examples.

2.1. Derivation of the Fourier Coefficients

We now have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Take the inner product of both sides with respect to the basis element 1.

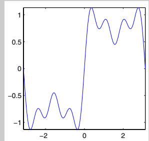
$$\langle f, 1\rangle = \langle (a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx), 1\rangle$$

Now, using the properties of the inner product,

$$\langle f, 1\rangle = a_0\langle 1, 1\rangle + \sum_{n=1}^{\infty}(a_n\langle \cos nx, 1\rangle + b_n\langle \sin nx, 1\rangle)$$

But, due to the orthogonality between the basis elements, all but one of the terms go to zero. Thus, we are left with

$$\langle f, 1\rangle = a_0\langle 1, 1\rangle.$$



Recall that we established $\langle 1, 1\rangle = \pi$. Therefore,

$$\begin{aligned} \langle f, 1\rangle &= a_0(2\pi), \\ a_0 &= \frac{1}{2\pi}\langle f, 1\rangle, \\ a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx. \end{aligned}$$

Now, take the inner product of both sides with respect to the basis element $\cos nx$.

$$\begin{aligned} \langle f, \cos nx\rangle &= \langle (a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx), \cos nx\rangle \\ \langle f, \cos nx\rangle &= a_0\langle 1, \cos nx\rangle + \sum_{n=1}^{\infty}(a_n\langle \cos nx, \cos nx\rangle + b_n\langle \sin nx, \cos nx\rangle) \end{aligned}$$

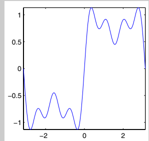
Again, the orthogonality of the elements simplifies this equation to

$$\langle f, \cos nx\rangle = a_n\langle \cos nx, \cos nx\rangle.$$

But, we previously found the value of $\langle \cos nx, \cos nx\rangle$.

$$\begin{aligned} \langle f, \cos nx\rangle &= a_n(\pi) \\ a_n &= \frac{1}{\pi}\langle f, \cos nx\rangle \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \end{aligned}$$

Finally, repeat the procedure taking the inner product of both sides with respect to



$\sin nx$, the remaining basis element.

$$\langle f, \sin nx\rangle = \langle (a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx), \sin nx\rangle$$

$$\langle f, \sin nx\rangle = a_0\langle 1, \sin nx\rangle + \sum_{n=1}^{\infty}(a_n\langle \cos nx, \sin nx\rangle + b_n\langle \sin nx, \sin nx\rangle)$$

Remember, $\sin nx$ is orthogonal to every basis element but itself.

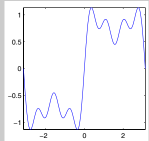
$$\begin{aligned} \langle f, \sin nx\rangle &= b_n\langle \sin nx, \sin nx\rangle \\ \langle f, \sin nx\rangle &= b_n\pi \\ b_n &= \frac{1}{\pi}\langle f, \sin nx\rangle \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \end{aligned}$$

2.2. Using the Fourier Series

To demonstrate the Fourier series, we will expand the function

$$f(x) = x, \quad -\pi \le x \le \pi.$$

See Figure 4. We being by finding the coefficients. Start with a_0. If we think graphically, a_0 represents the initial weight or displacement. We should look at the area under this curve. The same amount of area exists to the left and right of the y-axis. But, the area on the left is negative, so our initial area under the curve, a_0 is zero. To prove it, the



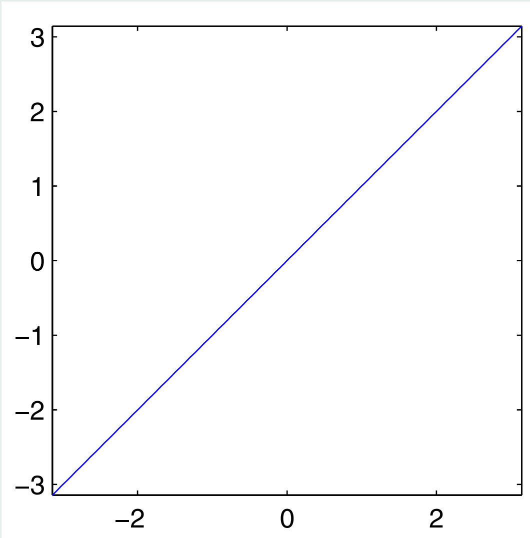
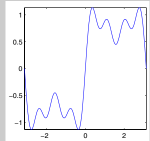


Figure 4: The graph of $f(x) = x$ from $[-\pi, \pi]$.

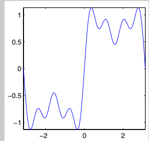


expression for a_0 is evaluated from $[-\pi, \pi]$.

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx \\
&= \frac{1}{2\pi}\left[x^2\right|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}[\pi^2 - (-\pi)^2] \\
&= \frac{1}{2\pi}(\pi^2 - \pi^2) \\
&= \frac{1}{2\pi}(0) \\
&= 0
\end{aligned}
$$

Now, we will calculate a_n.

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{1}{n}x\sin nx\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin nx\,dx \\
&= \frac{1}{n\pi}\left[x\sin nx\right|_{-\pi}^{\pi} + \frac{1}{n\pi}\left[\frac{1}{n}\cos nx\right|_{-\pi}^{\pi} \\
&= \frac{1}{n\pi}(\pi\sin n\pi - \pi\sin(-n\pi)) + \frac{1}{n^2\pi}\left[\cos nx\right|_{-\pi}^{\pi} \\
&= \frac{1}{n\pi}(0-0) + \frac{1}{n^2\pi}(\cos n\pi - \cos(-n\pi)) \\
&= 0 + \frac{1}{n^2\pi}(1-1) \\
&= 0
\end{aligned}
$$



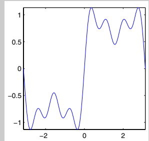
If we think of this problem graphically, we notice that $f(x) = x$ is an odd function, so cosine would not have any terms because it is an even function.

Finally, we will find b_n.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \\
&= \frac{1}{\pi}\left[-\frac{1}{n} x \cos nx \Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} \cos nx \, dx \right. \\
&= -\frac{1}{n\pi}\left[x \cos nx \right|_{-\pi}^{\pi} + \frac{1}{n\pi}\left[\frac{1}{n\pi} \sin nx \right|_{-\pi}^{\pi} \\
&= -\frac{1}{n\pi}\left(\pi \cos n\pi + \pi \cos(-n\pi) \right) + \frac{1}{n^2\pi}\left[\sin nx \right|_{-\pi}^{\pi} \\
&= -\frac{1}{n\pi}(2\pi) \cos n\pi + \frac{1}{n^2\pi}(\sin n\pi - \sin(-n\pi)) \\
&= -\frac{2\pi}{n\pi} \cos n\pi + \frac{1}{n^2\pi}(0) \\
&= -\frac{2}{n}(-1)^{n+1}
\end{aligned}
$$

Combining these three values into the Fourier series, we get

$$
\begin{aligned}
f(x) = x = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), &\quad -\pi \leq x \leq \pi \\
= 0 + \sum_{n=1}^{\infty}\left(0 \cos(nx) + \left(-\frac{2}{n}(-1)^{n+1} \right) \sin nx \right), &\quad -\pi \leq x \leq \pi \\
= 2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n} \sin nx \right), &\quad -\pi \leq x \leq \pi.
\end{aligned}
$$



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Expand this summation a few terms,

$$f(x) = x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right), \quad -\pi \leq x \leq \pi.$$

Notice the signs alternate and the frequency of each term is equal to the denominator of each term.

Now, we will graph the summation with an increasing number of terms. We start with four terms. The equation for Figure 5 follows.

$$f(x) = x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4}\right), \quad -\pi \leq x \leq \pi.$$

As we increase the number of terms in our series to ten, we see the improvement in the graph (see Figure 6), and the separation between the original function and modeled function decrease. But the new graph still doesn't look like $f(x) = x$. The middle of the graph of fifty terms shown in Figure 7 looks very close. But, even at $n = 50$, there appears to be some error at the very edges of the graph. This "ringing" is known as Gibb's phenomenon.

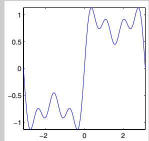
3. The Discrete Fourier Expansion

Our first derivation led to the Fourier coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \text{ and,}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx.$$



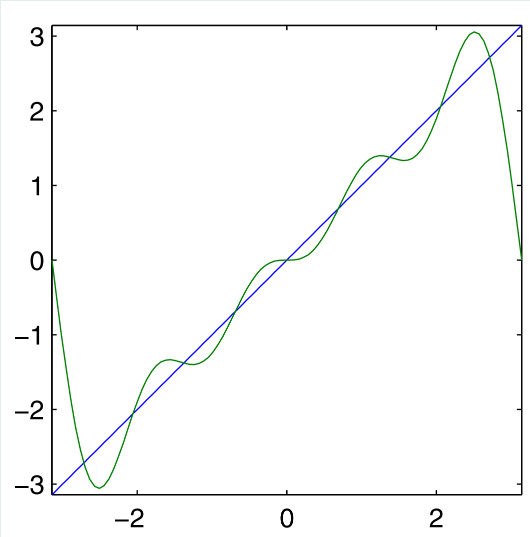
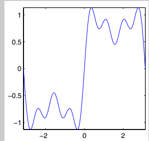


Figure 5: Here, $n = 4$.



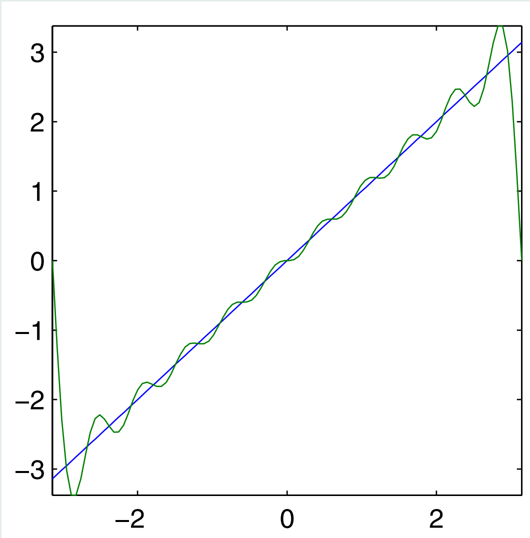
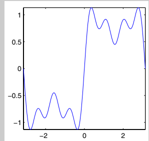


Figure 6: $n = 10$.



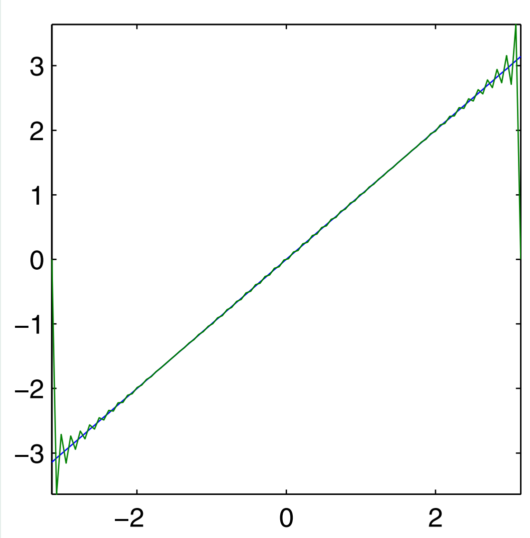
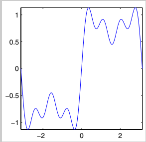


Figure 7: $n = 50$.



In the Fourier approximation of $f(x) = x$ on $[-\pi, \pi]$, we saw that equivalent representations for the Fourier were

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) d\,x,$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \text{ and,}$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

In general, the Fourier coefficients for the expansion of a function within period T are given by

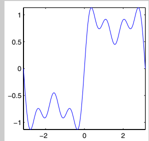
$$a_0 = \frac{1}{T} \int_{0}^{T} f(x)\, dx,$$
$$a_n = \frac{2}{T} \int_{0}^{T} f(x) \cos \frac{2\pi nx}{T}\, dx, \text{ and,}$$
$$b_n = \frac{2}{T} \int_{0}^{T} f(x) \sin \frac{2\pi nx}{T}\, dx$$

and the series is defined by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nx}{T} + b_n \sin \frac{2\pi nx}{T}.$$

If and when $f(x)$ is unknown or unable to be integrated, the Fourier series expansion doesn't work, but $f(x)$ can still be examined point by point with the Discrete Fourier Expansion. For example, consider the periodic wave in Figure 8. This is an analog sample that could represent a musical tone of some type or a periodic signal.

First, we will segregate a single period in Figure 9.



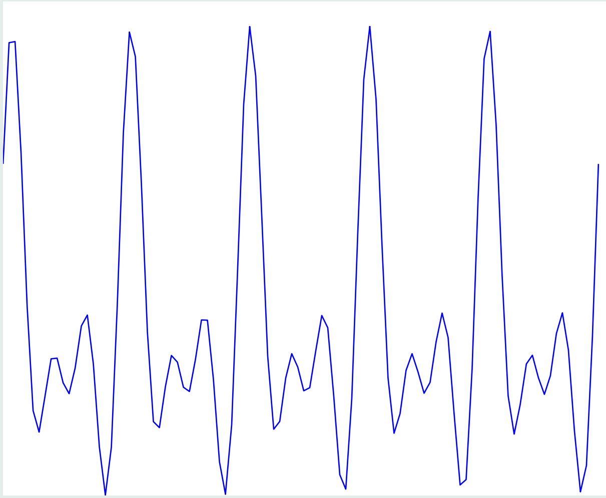
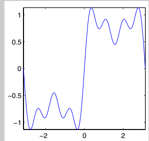


Figure 8: Periodic wave.



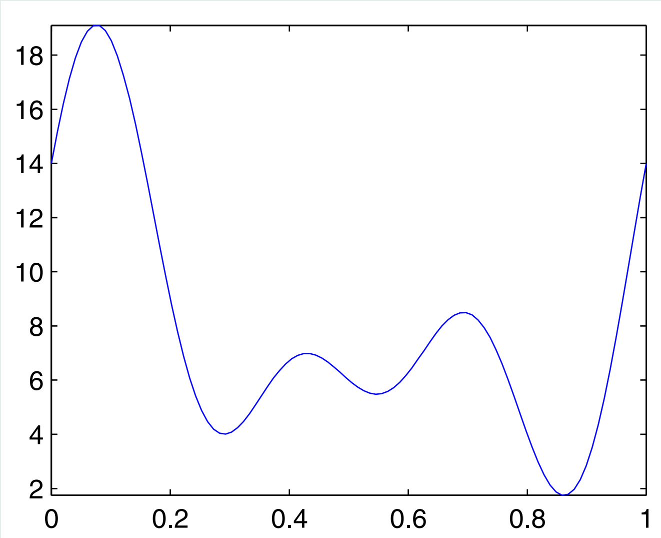
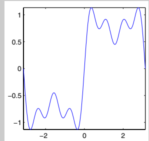


Figure 9: A single period of the wave.



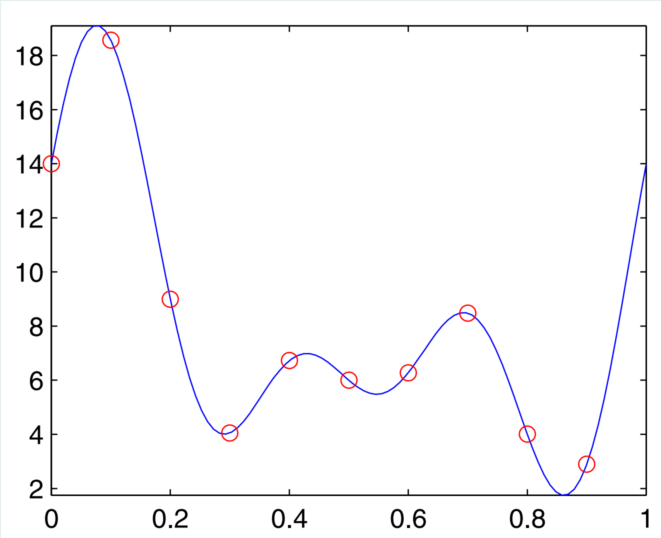


Figure 10: The left-endpoints used for the discrete values are circled in red.

Again, a_0 will be calculated first.

$$a_0 = \frac{1}{T} \int_0^T f(x)\, dx$$

The function of this single period is unknown, so instead of taking the integral over the period, we will use the left-endpoint method by selecting sample points to find the area under the curve. See Figure 10.

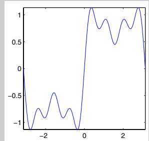
$$
\begin{aligned}
a_0 &\approx \frac{1}{T}\sum_{n=0}^{10-1} f(n \times \Delta x)\Delta x \\
&= \frac{1}{1}\sum_{n=0}^{9} f(n \times 0.1)0.1 \\
&= (f(0) + f(.1) + f(.2) + f(.3) + f(.4) + f(.5) \\
&\qquad + f(.6) + f(.7) + f(.8) + f(.9))(0.1) \\
&= (14.000 + 18.569 + 8.989 + 4.051 + 6.724 + 6.000 \\
&\qquad + 6.275 + 8.476 + 4.010 + 2.903)(0.1) \\
&= 8
\end{aligned}
$$

A more complete understanding of a_0 will send ambiguity forth to funerals. Cosine and sine waves osculate about the x-axis with a total sum of zero underneath their curves. In the Fourier series, a_0 behaves as a constant much like b in the formula $y = mx + b$. Every scaled sine and cosine wave (that makes up the character of the wave) will contain sums of zero underneath their curves. The addition of a_0 prevents a zero integral, provided a nonzero total area exists under the original function. Thusly, the value of a_0 is the area underneath the original and model wave.

Now,

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(x)\cos\frac{2\pi n x}{T}\,dx \\
a_1 &= \frac{2}{1}\int_0^1 f(x)\cos\frac{2\pi(1)x}{1}\,dx \\
&= 2\int_0^1 f(x)\cos(2\pi x)\,dx
\end{aligned}
$$

Again, we must use the left-endpoint method to find the area under the product curve



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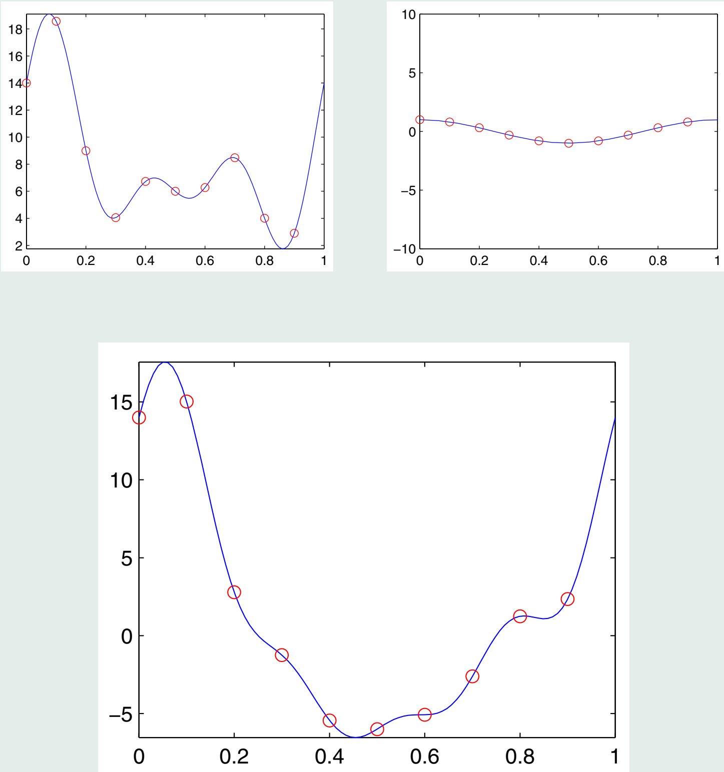
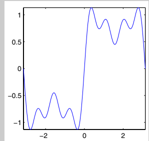


Figure 11: The original function (top left), the cosine (top right), and their product (bottom).



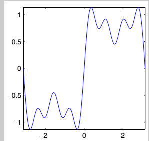
of $f(x)\cos 2\pi x$.

$$\begin{aligned}
a_1 &= 2(f(x)\cos(2\pi(0)) + f(.1)\cos(2\pi(.1)) + f(.2)\cos(2\pi(.2)) + \\
&\qquad \cdots + f(.9)\cos(2\pi(.9)))\Delta x \\
&= 2(14.000 - 15.023 + 2.778 - 1.252 - 5.440 - 6.000 \\
&\qquad - 5.077 - 2.619 + 1.239 + 2.349)(.1) \\
&= 3.022
\end{aligned}$$

We can store this information in a table.

		a_1	a_2	a_3	a_4	a_5
x	$f(x)$	$f(x)\cos\omega x$	$f(x)\cos 2\omega x$	$f(x)\cos 3\omega x$	$f(x)\cos 4\omega x$	$f(x)\cos 5\omega x$
0	14	14				
1	19	15.128				
2	9	2.781				
3	4.1	−1.267				
4	6.7	−5.420				
5	6	−6				
6	6.3	−5.096				
7	8.4	−2.569				
8	4	1.236				
9	2.9	2.346				
Total	area	15.112				
	a_n	3.0224				

Then we proceed to find a_2, a_3, a_4, and a_5 by summing the products of $f(x)\cos 2\pi nx$ (of their respective frequencies) and multiplying by Δx and $2/T$. Then, store this information into a table.

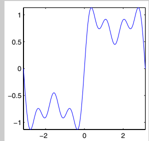


		a_1	a_2	a_3	a_4	a_5
x	$f(x)$	$f(x)\cos\omega x$	$f(x)\cos 2\omega x$	$f(x)\cos 3\omega x$	$f(x)\cos 4\omega x$	$f(x)\cos 5\omega x$
0	14	14	14	14	14	14
1	19	15.128	5.778	−5.778	−15.128	−18.7
2	9	2.781	−7.281	−7.281	2.781	9
3	4.1	−1.267	−3.317	3.317	1.267	−4.1
4	6.7	−5.420	2.070	2.070	−5.420	6.7
5	6	−6	6	−6	6	−6
6	6.3	−5.096	1.946	1.946	−5.097	6.3
7	8.4	−2.569	−6.796	6.796	2.596	−8.4
8	4	1.236	−3.236	−3.236	1.236	4
9	2.9	2.346	0.896	−0.896	−2.346	−2.9
Total	area	15.112	10.062	4.938	−0.112	−0.1
	a_n	3.0224	2.012	0.987	−0.022	−0.02

We can find b_1 in a similar manner.

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(x)\sin\frac{2\pi n x}{T}\,dx \\
b_1 &= \frac{2}{1}\int_0^1 f(x)\sin\frac{2\pi(1)x}{1}\,dx \\
b_1 &= 2\int_0^1 f(x)\sin 2\pi x\,dx \\
&\approx 2(f(0)\sin(2\pi(0)) + f(.1)\sin(2\pi(.1)) + f(.2)\sin(2\pi(.2)) + \cdots \\
&\qquad + f(.9)\sin(2\pi(.9)))(0.1) \\
&= 2.034
\end{aligned}
$$

The product of $f(x)$ and $\sin 2\pi x$ is demonstrated graphically in Figure 12. Discrete values from the product function are summed and then multiplied by $2/T$ and Δx to yield b_1.



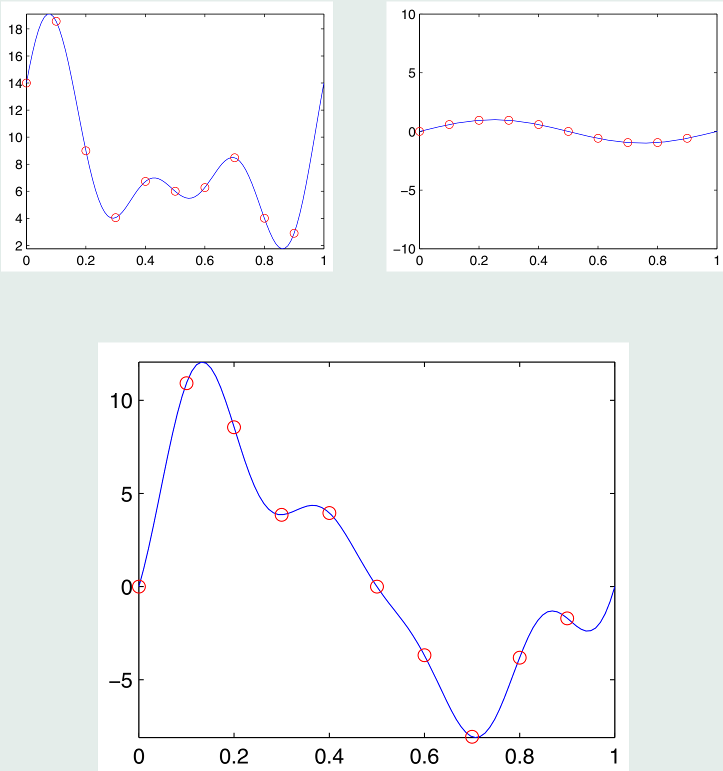
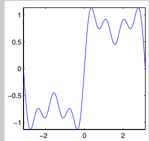


Figure 12: The original function (top left), the sine function (top right), and their product (bottom).



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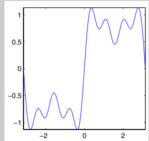
Then this information is placed in a table where the values of b_2, b_3, b_4, and b_5 are respectively placed as well.

		b_1	b_2	b_3	b_4	b_5
x	$f(x)$	$f(x)\sin\omega x$	$f(x)\sin 2\omega x$	$f(x)\sin 3\omega x$	$f(x)\sin 4\omega x$	$f(x)\sin 5\omega x$
0	14	0	0	0	0	0
1	19	10.992	17.786	17.786	10.992	0
2	9	8.560	5.290	−5.290	−8.560	0
3	4.1	3.900	−2.41	−2.41	3.900	0
4	6.7	3.938	−6.372	6.372	−3.938	0
5	6	0	0	0	0	0
6	6.3	−3.703	5.992	−5.992	3.073	0
7	8.4	−7.989	4.938	4.937	−7.989	0
8	4	-3.804	−2.351	2.351	3.804	0
9	2.9	-1.704	−2.758	−2.758	−1.704	0
Total	area	10.188	20.113	14.996	0.207	0
	b_n	2.038	4.023	2.999	0.041	0

Now, we can list the coefficients that were found discretely.

$$
\begin{aligned}
a_0 &= 8.000 & & \\
a_1 &= 3.022 & b_1 &= 2.038 \\
a_2 &= 2.012 & b_2 &= 4.023 \\
a_3 &= 0.988 & b_3 &= 2.999 \\
a_4 &= -0.022 & b_4 &= 0.041 \\
a_5 &= -0.020 & b_5 &= 0.000
\end{aligned}
$$



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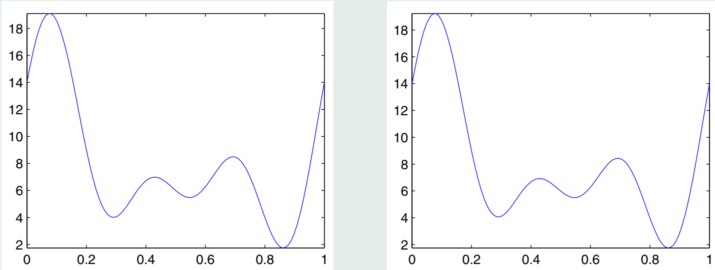


Figure 13: The original function (left) and our model with ten terms (right).

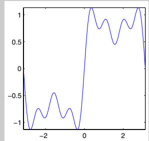
And, using these numbers, we can write

$$
\begin{aligned}
f(x) \approx\, & a_0 + a_1\cos(1\times 2\pi x) + b_1\sin(1\times 2\pi x + a_2\cos(2\times 2\pi x) \\
& + b_2\sin(2\times 2\pi x) + \ldots + a_5\cos(5\times 2\pi x) + b_5\sin(5\times 2\pi x), \\
=\, & 8 + 3.022\cos(2\pi x) + 2.038\sin(2\pi x) + 2.012\cos(4\pi x) \\
& + 4.023\sin(4\pi x) + 0.988\cos(6\pi x) + 2.999\sin(6\pi x) \\
& - 0.022\cos(8\pi x) + 0.041\sin(8\pi x) - 0.020\cos(10\pi x) \\
& + 0\sin(10\pi x).
\end{aligned}
$$

Now, we can compare the original and modeled functions in Figure 13 .

4. Interpreting Sound: An Application

Hearing is a complicated process that combines the structure of the ear with our newly understood Fourier coefficients. Sound waves are differences in air pressure through a period of time. The waves enter the ear through the ear canal and are channeled through



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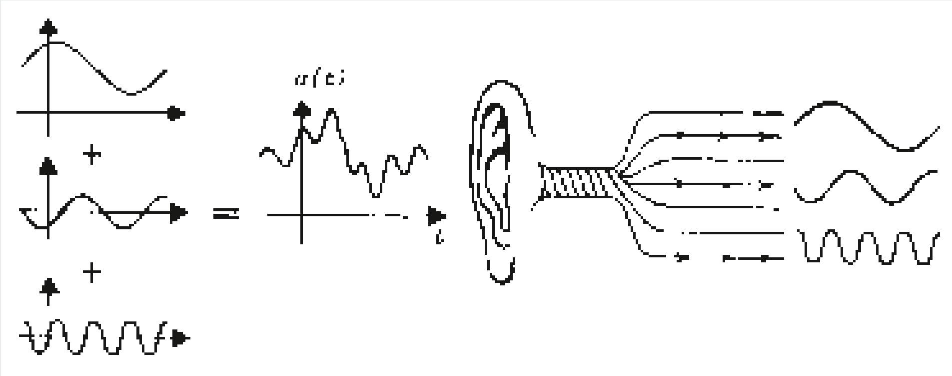
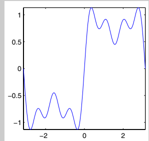


Figure 14: Three simple harmonic functions combine to make a “noise” that enters the ear. As the vibrations cause waves in the fluid of the cochlea, each cilia picks up the amplitude of a specific frequency and reports to the brain.

the outer ear to the ear drum. The ear drum’s movements vibrate the three tiny bones that connect the ear drum to the cochlea. The cochlea is a spiral shaped sensory organ that contains fluid and cilia, the little hair-like structures attached to nerve cells. While the ear drum is causing the connecting bones to move, those bones cause the fluid to oscillate. The cilia move with the fluid and send information to the nerve cells which pass it along to neurons, and finally the brain received the information. We must think of a noise as a combination of sinusoidal functions. These functions vary in frequency and in amplitude. Each cilia has an “assigned” frequency that causes it to respond to the fluid’s movement. The higher frequencies are located at the outer end of the cochlea while the lower frequencies are located towards the inner part. The amount the cilia vibrates depends on the amplitude from the wave of that specific frequency. The information that is reported to the brain is the amplitude.



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If we had a graph of a sound wave, we could use the Discrete Fourier Expansion to approximate the coefficients of the Fourier series. Then, by adding enough terms from the expansion, we could approximate the sound wave.

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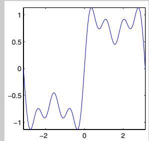

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