Pursuit Curves

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Assumptions

- At $t = 0$, merchant at $(x_0, 0)$, pirate at $(0, 0)$.
- Merchant's speed is V_m.
- Pirate's speed is V_p.
- Merchant travels along vertical line $x = x_0$.
- At time $t \geq 0$, pirate at (x, y).

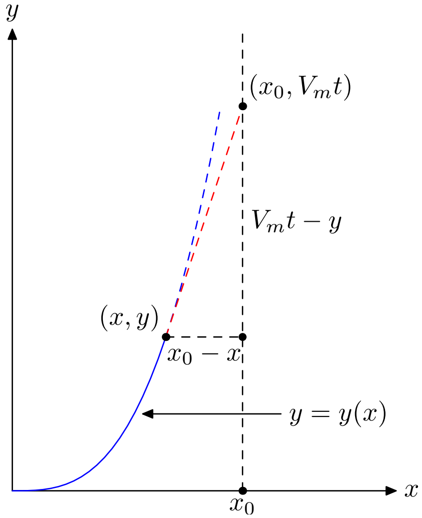

Figure: Geometry of pirate pursuit

$$\frac{dy}{dx} = \frac{V_m t - y}{x_0 - x}$$

$$V_p t = \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2}\, dz$$

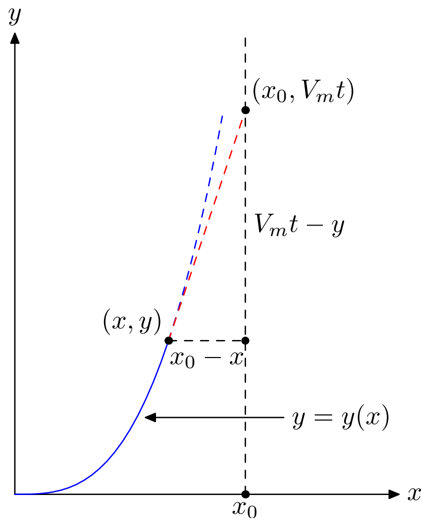

Figure: Geometry of pirate pursuit

Differential Equation for Pirate Pursuit

$$(x - x_0)\frac{dp}{dx} = -n\sqrt{1 + p^2(x)}$$

$$n = \frac{V_m}{V_p}\ ,\ p(x) = \frac{dy}{dx}$$

Separable Equation

$$\frac{dp}{\sqrt{1+p^2}} = \frac{-n\,dx}{x-x_0}$$

$$\ln(p+\sqrt{1+p^2}) + C = -n\ln(x_0 - x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(1-\frac{x}{x_0}\right)^{-n} - \left(1-\frac{x}{x_0}\right)^{n}\right]$$

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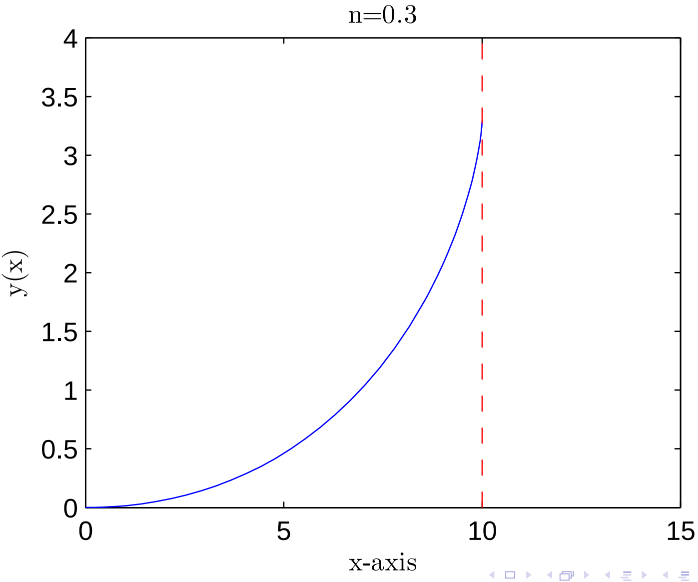
$$y(x) = \frac{1}{2}(x-x_0)\left[\frac{(1-x/x_0)^n}{1+n} - \frac{(1-x/x_0)^{-n}}{1-n}\right] + \frac{n}{1-n^2}x_0$$

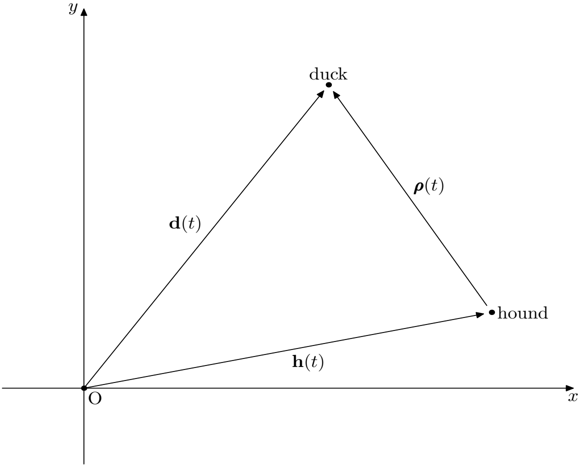
Results



Circular Pursuit

"A dog at the center of a circular pond makes straight for a duck which is swimming [counterclockwise] along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $n : 1$, determine the equation of the curve of pursuit..."

Generic Case



$$\mathbf{d}(t) = \mathbf{h}(t) + \boldsymbol{\rho}(t)$$

$$\mathbf{d}(t) = x_d(t) + iy_d(t) \qquad \mathbf{h}(t) = x_h(t) + iy_h(t)$$

Duck

- Duck's position vector given by

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Duck

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$$\mathbf{d}(t) = x_d(t) + iy_d(t)$$

- Duck's velocity vector given by

$$\frac{d\mathbf{d}(t)}{dt} = \frac{dx_d}{dt} + i\frac{dy_d}{dt}$$

Duck

- Duck's position vector given by

$$\mathbf{d}(t) = x_d(t) + iy_d(t)$$

- Duck's velocity vector given by

$$\frac{d\mathbf{d}(t)}{dt} = \frac{dx_d}{dt} + i\frac{dy_d}{dt}$$

- Duck's speed is

$$\left|\frac{d\mathbf{d}(t)}{dt}\right| = \sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2}$$

Hound

- Hound's position vector given by

$$\mathbf{h}(t) = x_h(t) + iy_h(t)$$

Hound

- Hound's position vector given by

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- Hound's velocity vector is given by

$$\frac{d\mathbf{h}(t)}{dt} = \left|\frac{d\mathbf{h}(t)}{dt}\right| \cdot \frac{\rho(t)}{|\rho(t)|} \tag{1}$$

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- Hound's speed is n times that of the duck,

$$\left|\frac{d\mathbf{h}(t)}{dt}\right| = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2}$$

- Equation (1) becomes

$$\frac{d\mathbf{h}(t)}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{\mathbf{d}(t) - \mathbf{h}(t)}{|\mathbf{d}(t) - \mathbf{h}(t)|}$$

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- In Cartesian Coordinates,

$$\frac{dx_h}{dt} + i\frac{dy_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{(x_d - x_h) + i(y_d - y_h)}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)^2}}$$

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- Equating real and imaginary parts leads to...

Equations for General Pursuit

$$\frac{dx_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{x_d - x_h}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)^2}}$$

$$\frac{dy_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{y_d - y_h}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)^2}}$$

- If the duck swims counterclockwise around a unit circle,

$$x_d(t) = \cos(t)\ , \qquad y_d(t) = \sin(t)$$

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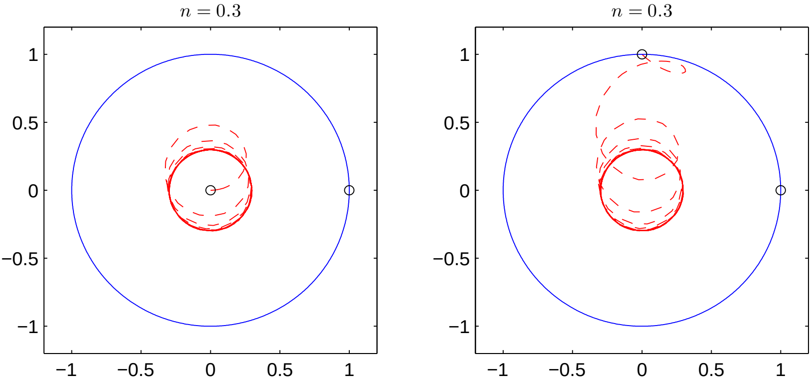
- Also,

$$n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} = n\sqrt{\sin^2(t) + \cos^2(t)} = n$$

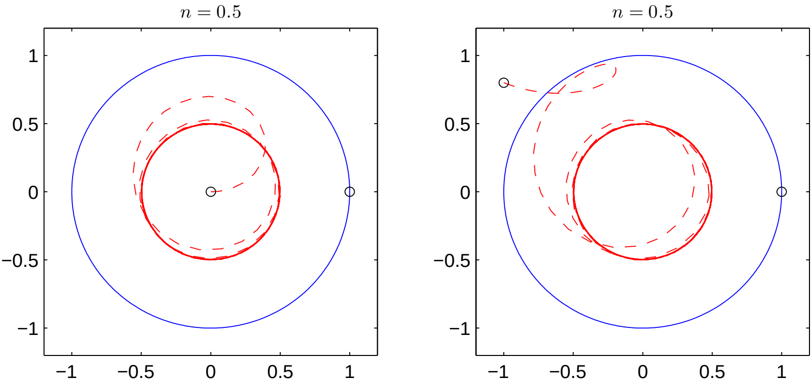
Circle Pursuit

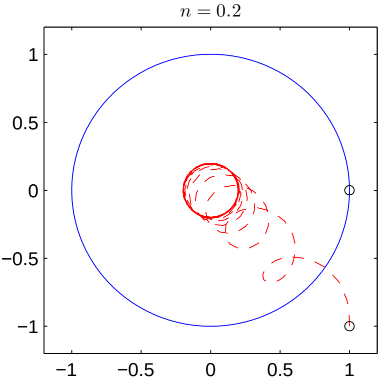
$$\frac{dx_h}{dt} = n\frac{\cos(t) - x_h}{\sqrt{(\cos(t) - x_h)^2 + (\sin(t) - y_h)^2}}$$

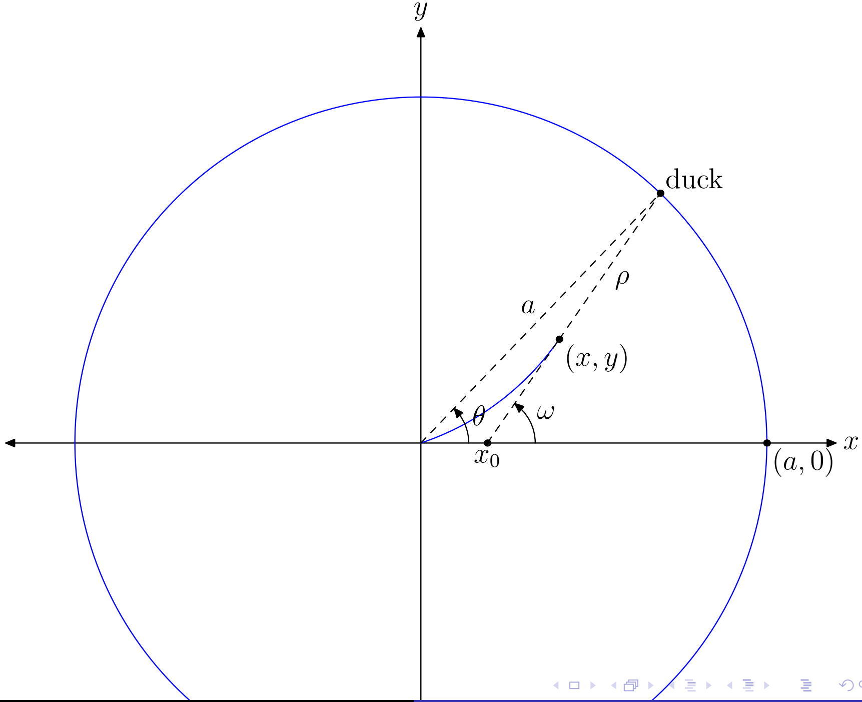
$$\frac{dy_h}{dt} = n\frac{\sin(t) - y_h}{\sqrt{(\cos(t) - x_h)^2 + (\sin(t) - y_h)^2}}$$



n = 0.3
n = 0.3
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1


$n = 0.5$
$n = 0.5$
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1


$n = 0.2$
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
$n = 0.7$
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1


y
x
duck
a
ρ
(x, y)
θ
ω
x0
(a, 0)

- Equation of tangent line:

$$y\cos(\omega) - x\sin(\omega) = -a\sin(\omega - \theta)$$

- Equation of normal line:

$$x\cos(\omega) + y\sin(\omega) = a\cos(\omega - \theta) - \rho$$

Differentiate tangent line

$$\frac{dx}{d\theta}\sin(\omega)-\frac{dy}{d\theta}\cos(\omega)+\frac{d\omega}{d\theta}[x\cos(\omega)+y\sin(\omega)] = a\cos(\omega-\theta)\left(\frac{d\omega}{d\theta}-1\right)$$

Differentiate tangent line

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$$\boxed{\rho\frac{d\omega}{d\theta} = a\cos(\omega-\theta)}$$

Differentiate normal line

$$\frac{dx}{d\theta}\cos(\omega) - x\sin(\omega)\frac{d\omega}{d\theta} + \frac{dy}{d\theta}\sin(\omega) + y\cos(\omega)\frac{d\omega}{d\theta}$$
$$= -a\sin(\omega-\theta)\left(\frac{d\omega}{d\theta} - 1\right) - \frac{d\rho}{d\theta}$$

Differentiate normal line

$$\frac{dx}{d\theta}\cos(\omega) - x\sin(\omega)\frac{d\omega}{d\theta} + \frac{dy}{d\theta}\sin(\omega) + y\cos(\omega)\frac{d\omega}{d\theta} = -a\sin(\omega-\theta)\left(\frac{d\omega}{d\theta}-1\right) - \frac{d\rho}{d\theta}$$

$$\boxed{\frac{d\rho}{d\theta} = a[\sin(\omega-\theta) - n]}$$

$$\boxed{\rho\frac{d\omega}{d\theta} = a\cos(\omega - \theta)} \quad \boxed{\frac{d\rho}{d\theta} = a[\sin(\omega - \theta) - n]}$$

$$\phi = \omega - \theta$$

$$\frac{d\omega}{d\theta} = \frac{d\phi}{d\theta} + 1$$

$$\rho\frac{d^2\rho}{d\theta^2} + a\rho\cos(\phi) = a^2\cos^2(\phi)$$

$$\frac{d\rho}{d\theta} = a\sin(\phi) - an$$

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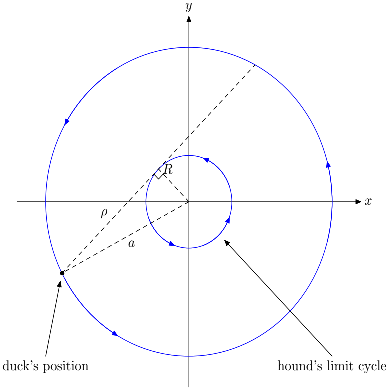
$$\frac{d\rho}{d\theta} = a\sin(\phi) - an$$



- $\lim_{\theta\to\infty} \rho = c$
- $\frac{d\rho}{d\theta} = \frac{d^2\rho}{d\theta^2} = 0$
- As $\theta \to \infty$, $\rho = a\cos(\phi)$
- As $\theta \to \infty$, $\sin(\phi) = n$

As $\theta \to \infty$...

$$a\rho\left(\frac{\rho}{a}\right) = a^2[1 - \sin^2(\phi)] = a^2(1 - n^2)$$

$$\lim_{\theta \to \infty} \rho = a\sqrt{1 - n^2}$$

The Limit Cycle

Letting R be the radius of the limit cycle,

$$R^2 + \rho^2 = a^2$$

$$R = na$$