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# Edgar Allan Poe The Pit and the Pendulum

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## Abstract

Edgar Allen Poe wrote some of the most original and terrifying horror stories of the 19th century. Poe's protagonists often met with grizzly deaths. His classic "The Pit and the Pendulum" is no exception to this. In Poe's story, a man is strapped to a table and forced to watch as a ever lengthening pendulum tipped with a scythe swings closer and closer to his chest. In this paper, we investigate not the psychological, but the physical aspects of a lengthening pendulum. We describe the lengthening pendulum in terms of Newtonian and Lagrangian mechanics and derive a differential equation for the pendulum's motion in terms of its angle. We then examine numerical methods of solving this differential equation and transform our differential equation into a Bessel equation to discover an analytical solution. Finally, we examine the motion of such a pendulum in reference to Poe's story. While our paper may not be as terrifying as a Poe tale (unless you regard intense applications of the chain rule terrifying), it is sure to be more informative.

"Down and still down it came -- to cross the region of the heart!..." POE

# 1. Introduction

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“Looking upward, I surveyed the ceiling of my prison. It was some thirty or forty feet overhead, and constructed much as the side walls. In one of its panels a very singular figure riveted my whole attention . It was the painted figure of Time as he is commonly represented, save that in lieu of a scythe he held what at a casual glance I supposed to be the pictured image of a huge pendulum, such as we see on antique clocks. There was something, however, in the appearance of this machine which caused me to regard it more attentively. While I gazed directly upward at it (for its position was immediately over my own), I fancied that I saw it in motion. In an instant afterward the fancy was confirmed. Its sweep was brief, and of course slow. I watched it for some minutes, somewhat in fear but more in wonder [...] It might have been half-an-hour, perhaps even an hour (for I could take but imperfect note of time) before I again cast my eyes upward. What I then saw confounded and amazed me. The sweep of the pendulum had increased in extent by nearly a yard. As a natural consequence, its velocity was also much greater. But what mainly disturbed me was the idea that it had perceptibly DESCENDED. I now observed, with what horror it is needless to say, that its nether extremity was formed of a crescent of glittering steel, about a foot in length from horn to horn; the horns upward, and the under edge evidently as keen as that of a razor. Like a razor also it seemed massy and heavy, tapering from the edge into a solid and broad structure above. It was appended to a weighty rod of brass, and the whole HISSED as it swung through the

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air [...] What boots it to tell of the long, long hours of horror more than mortal, during which I counted the rushing oscillations of the steel! Inch by inch – line by line – with a descent only appreciable at intervals that seemed ages – down and still down it came! Days passed – it might have been that many days passed – ere it swept so closely over me as to fan me with its acrid breath. [...] The vibration of the pendulum was at right angles to my length. I saw that the crescent was designed to cross the region of the heart. It would fray the serge of my robe; it would return and repeat its operations – again – and again. Notwithstanding its terrifically wide sweep (some thirty feet or more) and the hissing vigour of its descent, sufficient to sunder these very walls of iron, still the fraying of my robe would be all that, for several minutes, it would accomplish; and at this thought I paused. I dared not go farther than this reflection.”

This is the descending pendulum as Poe describes it. An ingeniously cruel torture device which starts with a sweep that is “brief, and [...]slow” but, which by the time it reaches its victim, possess a “terrifically wide sweep (some thirty feet or more) and the hissing vigour of its descent, sufficient to sunder these very walls of iron” and which swings through an arc so great that “the vibration of the pendulum was at right angles to [the victims] length” implying that the pendulum increases not only in sweep, but in its angle of motion, and in its speed. Such a device would truly be terrifying. We will attempt to model this lengthening pendulum and to discover if its real-life physical motion truly reflects Poe’s poetic description.

“Down and still down it came – to cross the region of the heart!...” POE



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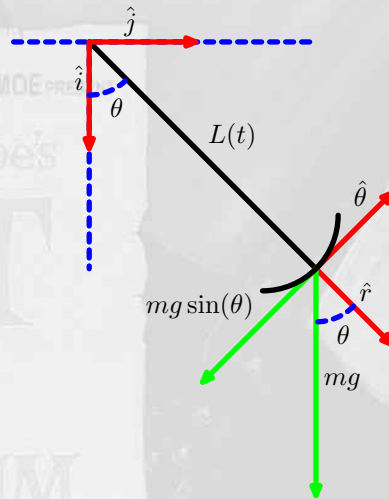


Figure 1: Poe's Pendulum Viewed in Terms of Newtonian Physics

## 2. Modeling Poe's Pendulum with Newtonian Mechanics

The main constraint of the problem is the changing length of the pendulum. Suppose that the length of the wire at time  $t$  is given as  $L(t)$ . We will let  $\theta(t)$  represent the angle that the wire of the pendulum makes with its resting vertical position. Suppose that  $\hat{r}$  and  $\hat{\theta}$  are unit vectors (see Figure 1),  $\hat{r}$  lying along the wire and  $\hat{\theta}$  always pointing in the direction of motion and orthogonal to the wire. Neglecting the damping force because it is minimal, and the tensile force of the

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wire because of its irrelevant direction, we shall only concern ourselves with forces directed along  $\hat{\theta}$ . We can obtain an equation of motion for the pendulum in the  $\hat{\theta}$  directions, but we must first do some preliminary work.

Let the position vector of the pendulum bob be represented as  $\vec{R} = L\hat{r}$ . It can now be shown that,

$$\begin{aligned}\hat{r} &= \cos \theta \hat{j} + \sin \theta \hat{i} \\ \hat{\theta} &= \cos \theta \hat{i} - \sin \theta \hat{j} \\ \hat{i} &= \sin \theta \hat{r} + \cos \theta \hat{\theta} \\ \hat{j} &= \cos \theta \hat{r} - \sin \theta \hat{\theta}.\end{aligned}$$

We now must use the chain rule to solve for  $\hat{r}'$  and  $\hat{\theta}'$ .

$$\hat{r}' = \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta} \frac{d\theta}{dt} = (-\sin \theta \hat{j} + \cos \theta \hat{i})\theta'$$

Thus,

$$\hat{r}' = \hat{\theta}\theta'. \quad (1)$$

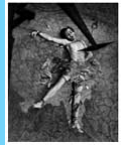
And,

$$\hat{\theta}' = \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \frac{d\theta}{dt} = (-\sin \theta \hat{i} - \cos \theta \hat{j})\theta'.$$

So,

$$\hat{\theta}' = -\hat{r}\theta'. \quad (2)$$

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Now that we have  $\hat{r}'$  and  $\hat{\theta}'$  we will look for the velocity  $\vec{R}'$  and acceleration  $\vec{R}''$  of the pendulum in terms of these functions, remembering that  $L(t)$  is a function of time, so we may not simply take it as a constant during differentiation.

$$v(t) = \hat{R}' = (L\hat{r})' = L\hat{r}' + \hat{r}L'.$$

Using (1) we get

$$\vec{R}' = L\hat{\theta}\theta' + \hat{r}L'.$$

Now,

$$\vec{R}'' = (L\hat{\theta}\theta' + \hat{r}L')'.$$

Thus,

$$\vec{R}'' = L\hat{\theta}\theta'' + L\hat{\theta}'\theta' + L'\hat{\theta}\theta' + \hat{r}L'' + L'\hat{r}'.$$

Using (2) we can rewrite as

$$\vec{R}'' = L\hat{\theta}\theta'' - L(-\theta'\hat{r})\theta' + L'\hat{\theta}\theta' + \hat{r}L'' + L'\hat{r}'.$$

Using (1) we get,

$$\vec{R}'' = L\hat{\theta}\theta'' - L(-\theta'\hat{r})\theta' + L'\hat{\theta}\theta' + \hat{r}L'' + L'\theta'\hat{\theta},$$

and after grouping like terms we are left with,

$$\vec{R}'' = (L'' - L\theta'\theta')\hat{r} + (L\theta'' + 2L'\theta')\hat{\theta}.$$

Which is our acceleration vector expressed as components of  $\hat{r}$  and  $\hat{\theta}$ .

We are now in position to implement Newton's second law. Notice that in our simplified model the only force acting on the pendulum is the component of the



gravitational force, which is opposite to the direction of motion and written as  $-mg \sin \theta$ . If the swinging blade has mass  $m$ , then we can say

$$F = ma.$$

Letting  $a$  be the component of acceleration along  $\hat{\theta}$  we get the equation of angular motion,

$$-mg \sin \theta = m(L\theta'' + 2L'\theta').$$

Canceling the mass on both sides and setting this equation to zero gives us the equation

$$= L\theta'' + 2L'\theta' + g \sin \theta \quad (3)$$

and making the assumption that if we use small angles,  $\sin \theta = \theta$ , we have our preliminary linear equation of motion for the swinging blade

$$L\theta'' + 2L'\theta' + g\theta = 0. \quad (4)$$

We now have an equation that describes the motion of the pendulum in terms of its length and its angle. As we mentioned earlier,  $L$  is function of time and at this point we will make our next assumption. Let's assume that the pendulum is descending at a constant velocity  $b$ . This means that at any time  $t$  the length  $L$  of the pendulum can be expressed as

$$L(t) = a + bt$$

where  $a$  is an arbitrary initial length.

Combining this result with (4) we have our equation for the angular motion of the pendulum.

$$(a + bt)\theta'' + 2b\theta' + g\theta = 0. \quad (5)$$

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### 3. Modeling Poe's Pendulum Revisited Using Lagrangian Mechanics

We do not need to use Newtonian physics to find equation (3). We can also use Lagrangian mechanics. Lagrangian mechanics looks at motion in terms of energy rather than forces as in classical Newtonian mechanics. A full explanation of Lagrangian mechanics is beyond the scope of this paper, but the mathematics of Lagrangian mechanics are fairly straight forward and can easily be used to find our equation of angular motion. For our purposes, Lagrangian mechanics revolves around two equations. The first is called the Lagrangian.

The Lagrangian is defined as

$$\text{Lagrangian}(L) = \text{Kinetic Energy}(KE) - \text{Potential Energy}(PE) \quad (6)$$

where,

$$KE = \frac{1}{2}mV^2$$

$$PE = \text{height} \times mg.$$

Since L is already being used for the Lagrangian, we will use R for the length of our pendulum cable remembering that R is a function of time. For our system,

$$KE = \frac{1}{2}m(R\theta')^2 \quad (7)$$

because the linear velocity of the pendulum is  $R\theta'$ . Looking at Figure 2, we can see that the height of the pendulum above the ground is  $d - R \cos(\theta)$  and thus,

$$PE = mg(d - R \cos(\theta)). \quad (8)$$



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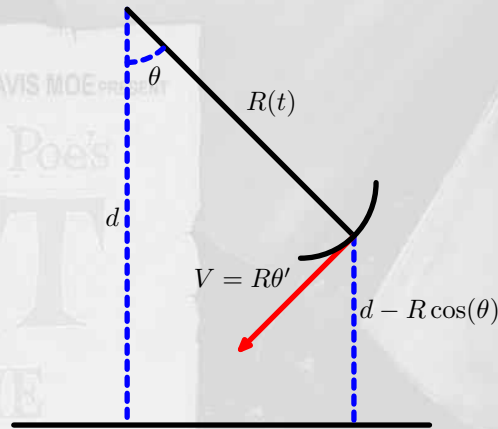


Figure 2: The Pendulum Viewed in Terms of the Lagrangian

Inserting equations (7) and (8) into the Lagrangian, we find that for our system

$$L = \frac{1}{2}m(R\theta')^2 - mg(d - R\cos(\theta)).$$

The second important equation in Lagrangian mechanics is Lagrange's equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial Q'} \right) = \frac{\partial L}{\partial Q}$$

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where  $Q$  is the function that is being solved for. We are solving for  $\theta$ , and thus,

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \theta'} \right) &= \frac{\partial L}{\partial \theta} \\ \frac{d}{dt} (mR^2 \theta') &= -mgR \sin(\theta) \\ 2mR R' \theta' + \theta'' mR^2 &= -mgR \sin(\theta). \end{aligned}$$

Canceling  $mR$  from both sides and setting the equation to zero we find

$$R\theta'' + 2R'\theta' + g \sin(\theta) = 0$$

which is our equation of angular motion (equation (3)). If we make the substitution  $\sin(\theta) = \theta$  for small values of  $\theta$ , then our equation becomes

$$R\theta'' + 2R'\theta' + g\theta = 0$$

which is equation (4).

## 4. A System of Equations

We can rewrite equation (3) as a system of equations. We start by writing

$$\begin{bmatrix} u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} \theta \\ \theta' \end{bmatrix}$$

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and

$$\begin{bmatrix} u(1)' \\ u(2)' \end{bmatrix} = \begin{bmatrix} \theta' \\ \theta'' \end{bmatrix}$$

therefore,

$$\begin{bmatrix} u(1)' \\ u(2)' \end{bmatrix} = \begin{bmatrix} u(2) \\ -\frac{2L'u(2)}{L} - \frac{g \sin(u(1))}{L} \end{bmatrix}$$

We can solve this system in MATLAB using ODE45. MATLAB will solve our differential equation numerically and give us results  $u(1) = \theta$  and  $u(2) = \theta'$ . Letting  $L(t) = a + bt$  and solving the system with using  $a = 1$  m,  $b = 0.1$  m/s,  $g = 9.8$  m/s<sup>2</sup> (normal gravity on earth),  $\theta_0 = \pi/2$  (we start the pendulum at horizontal), and  $\theta' = 0$  (we start the pendulum at rest) and using this data to plot  $\theta$  vs. time produces the results seen in Figure 3. We can see that the angle decays over time. However, if we look at the sweep or arc length ( $L\theta$ ) traveled by the pendulum, we notice that it actually increases. From this, we can come to understand that the angle decays at a slower rate than the length of the pendulum cable increases. Next, we will look at the change in the angular velocity  $\theta'$ . We notice that this also rather sharply decays with time. If we look at the linear velocity of the pendulum  $L\theta'$ , we notice that it too decreases with time although not as sharply. The decay in the linear velocity is tempered by the increasing pendulum cable length. Finally, if we look at a plot of the path of the pendulum (Figure 4), we can see the change in sweep and angular velocity all in a single picture. The increasing sweep is evident from the widening path of the pendulum, and the decreasing angular velocity is apparent from the ever increasing period of the pendulum.

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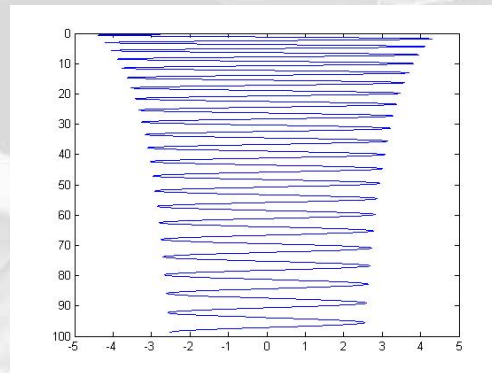
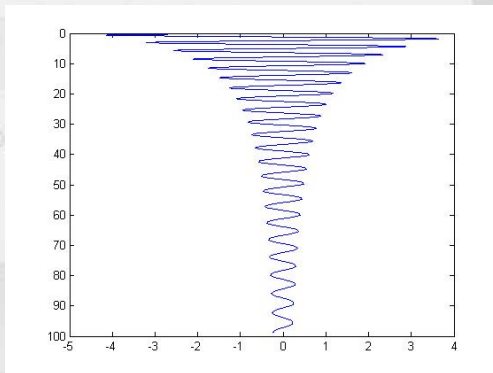
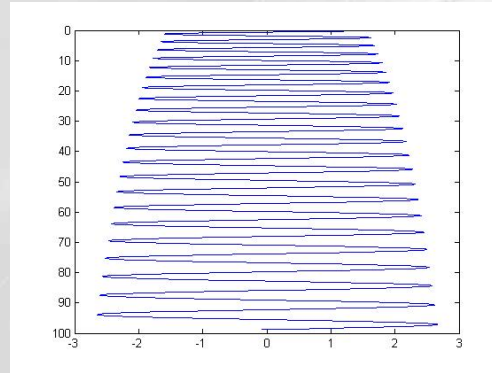
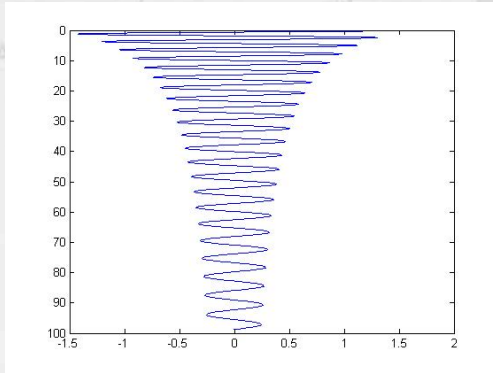
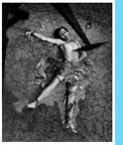


Figure 3:  
From top left clockwise: 1. The Angle, 2. The Sweep,  
3. The Linear Velocity, 4. The Angular Velocity.

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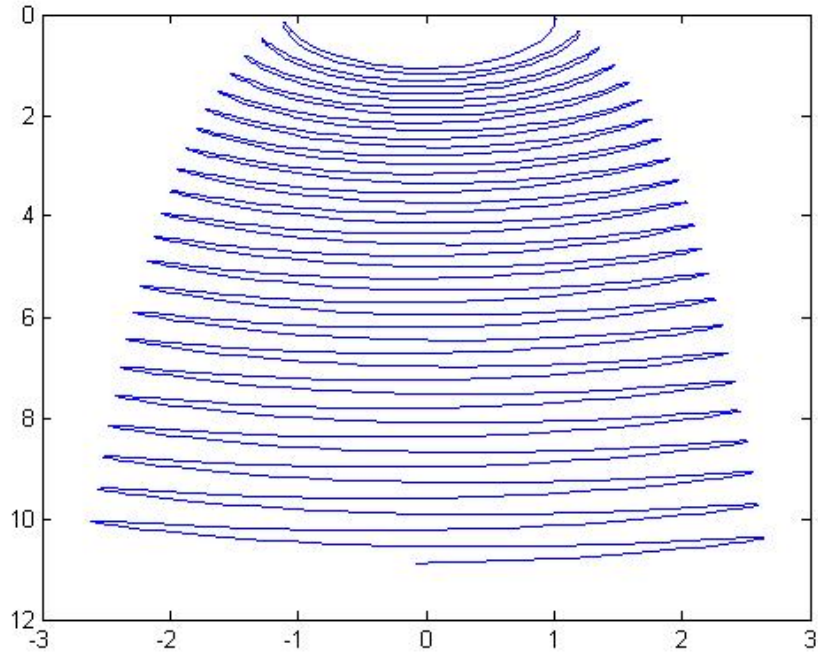


Figure 4: The Path of the Pendulum

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## 5. A Change of Variables

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We now have a numerical approximation of the motion of our pendulum, but it would be more satisfying to develop an analytical model. The model we have (5) is not directly solvable. However, if one is lucky, one may realize that it can be transformed, using a change of variables, into a Bessel equation of the form

$$\frac{d^2y}{dx^2}x^2 + x\frac{dy}{dx} + y(x^2 - 1) = 0. \quad (9)$$

We will start by introducing the change of variable

$$x = \frac{2}{b}\sqrt{a + bt}\sqrt{g}.$$

Therefore,

$$\sqrt{a + bt} = \frac{xb}{2\sqrt{g}} \quad (10)$$

$$a + bt = \frac{x^2b^2}{4g}. \quad (11)$$

We will also introduce the change of variable

$$y = \theta\sqrt{a + bt}.$$

Solving this for  $\theta$  we find

$$\theta = \frac{y}{\sqrt{a + bt}}.$$

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Substituting equation (10) into this, we obtain

$$\theta = \frac{2y\sqrt{g}}{bx}.$$

To transform (5) into our Bessel equation, we need to find  $\theta'$  and  $\theta''$  in terms of our new variables  $y$  and  $x$ . We will first find  $\theta'$  using an application of the chain and product rules.

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d}{dt} \left( \frac{2y\sqrt{g}}{bx} \right) \\ &= \frac{2\sqrt{g}}{b} \frac{d}{dt} \left( y \frac{1}{x} \right) \\ &= \frac{2\sqrt{g}}{b} \left( \frac{dy}{dt} \frac{1}{x} + \frac{d}{dt} \left( \frac{1}{x} \right) y \right) \\ &= \frac{2\sqrt{g}}{b} \left( \frac{dy}{dx} \frac{dx}{dt} \frac{1}{x} + \frac{d}{dx} \left( \frac{1}{x} \right) \frac{dx}{dt} y \right).\end{aligned}$$

Fortunately for us at this point, we can calculate  $dx/dt$ .

$$\frac{dx}{dt} = \frac{\sqrt{g}}{\sqrt{a+bt}}$$

Substituting equation (10) into the above results in

$$\frac{dx}{dt} = \frac{2g}{bx}.$$

“Down and still  $\frac{dx}{dt} = \frac{2g}{bx}$  time – to cross the region of the heart!...” POE



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Thus, continuing our computation of  $\theta'$  we find

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{2\sqrt{g}}{b} \left( \frac{dy}{dx} \frac{2g}{bx} \frac{1}{x} - \frac{1}{x^2} \frac{2g}{bx} y \right) \\ &= \frac{4\sqrt{g}g}{b^2} \left( \frac{dy}{dx} \frac{1}{x^2} - \frac{1}{x^3} y \right).\end{aligned}$$

From this, we will find  $\theta''$  using similar methods and, again, our substitution for  $\frac{dx}{dt}$ , as follows.

$$\begin{aligned}\frac{d^2\theta}{dt^2} &= \frac{4\sqrt{g}g}{b^2} \frac{d}{dt} \left( \frac{dy}{dx} \frac{1}{x^2} - \frac{1}{x^3} y \right) \\ &= \frac{4\sqrt{g}g}{b^2} \left[ \left( \frac{d}{dt} \left( \frac{dy}{dx} \right) \right) \frac{1}{x^2} + \frac{d}{dt} \left( \frac{1}{x^2} \right) \frac{dy}{dx} \right] - \left( \frac{d}{dt} \left( \frac{1}{x^3} \right) y + \frac{dy}{dt} \frac{1}{x^3} \right) \\ &= \frac{4\sqrt{g}g}{b^2} \left[ \left( \frac{d}{dx} \left( \frac{dy}{dx} \right) \right) \frac{dx}{dt} \frac{1}{x^2} + \frac{d}{dx} \left( \frac{1}{x^2} \right) \frac{dx}{dt} \frac{dy}{dx} \right] - \left( \frac{d}{dx} \left( \frac{1}{x^3} \right) \frac{dx}{dt} y + \frac{dy}{dx} \frac{dx}{dt} \frac{1}{x^3} \right) \\ &= \frac{4\sqrt{g}g}{b^2} \left( \frac{d^2y}{dx^2} \frac{2g}{bx} \frac{1}{x^2} - \frac{2}{x^3} \frac{2g}{bx} \frac{dy}{dx} + \frac{3}{x^4} \frac{2g}{bx} y - \frac{dy}{dx} \frac{2g}{bx} \frac{1}{x^3} \right) \\ &= \frac{8\sqrt{g}g^2}{b^3x^5} \left( \frac{d^2y}{dx^2} x^2 - 3x \frac{dy}{dx} + 3y \right).\end{aligned}$$

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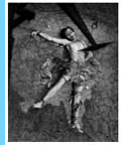
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Placing our newly found equations for  $\theta'$  and  $\theta''$  into equation (5), we find

$$\begin{aligned}
 (a + bt) \left[ \frac{8\sqrt{g}g^2}{b^3x^5} \left( \frac{d^2y}{dx^2}x^2 - 3x \frac{dy}{dx} + 3y \right) \right] + 2b \left[ \frac{4\sqrt{g}g}{b^2} \left( \frac{dy}{dx} \frac{1}{x^2} - \frac{1}{x^3}y \right) \right] + g \frac{y}{\sqrt{a + bt}} &= 0 \\
 \frac{x^2b^2}{4g} \left[ \frac{8\sqrt{g}g^2}{b^3x^5} \left( \frac{d^2y}{dx^2}x^2 - 3x \frac{dy}{dx} + 3y \right) \right] + \frac{8\sqrt{g}g}{b^3} \left( \frac{dy}{dx} \frac{1}{x^2} - \frac{1}{x^3}y \right) + \frac{2g\sqrt{g}}{xb}y &= 0 \\
 \frac{2\sqrt{g}g}{bx^3} \left( \frac{d^2y}{dx^2}x^2 - 3x \frac{dy}{dx} + 3y \right) + \frac{8\sqrt{g}g}{b^3} \left( \frac{dy}{dx} \frac{1}{x^2} - \frac{1}{x^3}y \right) + \frac{2g\sqrt{g}}{xb}y &= 0 \\
 \frac{2\sqrt{g}g}{bx^3} \left[ \left( \frac{d^2y}{dx^2}x^2 - 3x \frac{dy}{dx} + 3y \right) + 4 \left( x \frac{dy}{dx} - y \right) + x^2y \right] &= 0 \\
 \frac{2\sqrt{g}g}{bx^3} \left( \frac{d^2y}{dx^2}x^2 + x \frac{dy}{dx} - y + x^2y \right) &= 0 \\
 \frac{d^2y}{dx^2}x^2 + x \frac{dy}{dx} + y(x^2 - 1) &= 0.
 \end{aligned}$$

which is our desired Bessel equation (9).

## 6. Finding Initial Conditions for our Bessel Function from $\Theta$ and Time

In order to perform any meaningful calculations using our Bessel, we need to find initial conditions for  $x$  and  $y$  from our initial conditions for  $t$  and  $\theta$ . We can do this using our previously defined change of variables.

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Using our change of variable for  $x$

$$x = \frac{2}{b}\sqrt{a+bt}\sqrt{g},$$

it is easy to see that

$$x_{\text{initial}} = \frac{2}{b}\sqrt{a+bt_{\text{initial}}}\sqrt{g}$$

and likewise,

$$x_{\text{final}} = \frac{2}{b}\sqrt{a+bt_{\text{final}}}\sqrt{g}.$$

Using our change of variable for  $y$

$$y = \theta\sqrt{a+bt},$$

we can equally easily find our initial value for  $y$  at  $t_{\text{initial}}$

$$y_{\text{initial}} = \theta\sqrt{a+bt_{\text{initial}}}.$$

We next wish to find  $y'$  at  $t_{\text{initial}}$  but  $y$  is a function of  $x$  not  $t$ . Thus doing so requires another application of the chain rule.

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{d}{dt}(\theta\sqrt{a+bt}) \frac{dt}{dx} \\ &= \left( \theta' \sqrt{a+bt} + \frac{\theta b}{2\sqrt{a+bt}} \right) \frac{bx}{2g}. \end{aligned}$$

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Therefore, we find  $y_{\text{initial}}$  to be

$$\begin{aligned}y'_{\text{initial}} &= \left( \theta'(t_{\text{initial}}) \sqrt{a + bt_{\text{initial}}} + \frac{\theta(t_{\text{initial}})b}{2\sqrt{a + bt_{\text{initial}}}} \right) \frac{bx_0}{2g} \\ &= \left( \theta'(t_{\text{initial}}) \sqrt{a + bt_{\text{initial}}} + \frac{\theta(t_{\text{initial}})b}{2\sqrt{a + bt_{\text{initial}}}} \right) \frac{\sqrt{a + bt_{\text{initial}}}}{\sqrt{g}} \\ &= \frac{\theta'(t_{\text{initial}})(a + bt_{\text{initial}})}{\sqrt{g}} + \frac{\theta(t_{\text{initial}})b}{2\sqrt{g}}.\end{aligned}$$

If we start our analysis at  $t = 0$ , then the equation simplifies to

$$y'_{\text{initial}} = \frac{\theta'(0)a}{\sqrt{g}} + \frac{\theta(0)b}{2\sqrt{g}}.$$

If we release our pendulum from rest, then  $\theta' = 0$  and the equation further simplifies to

$$y'_{\text{initial}} = \frac{\theta(0)b}{2\sqrt{g}}.$$

We also need to be able to convert  $x$  and  $y$  back into  $t$  and  $\theta$ . Using  $a + bt = x^2b^2/4g$ , we find

$$t = \frac{x^2b}{4g} - \frac{a}{b}, \quad (12)$$

and substituting this into our equation for  $\theta$  we find

$$\theta = \frac{y}{\sqrt{a + b \left( \frac{x^2b}{4g} - \frac{a}{b} \right)}}. \quad (13)$$

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## 7. Computing and analyzing the Bessel

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We have thus far only mentioned the Bessel equation as the form of our equation of motion. But what is a Bessel equation? Our goal here will be to briefly present the Bessel equation by only skimming the mathematical prerequisites.

Bessel equations of order  $r$  as we have seen, take the form

$$x^2 y'' + xy' + (x^2 - r^2)y = 0, \quad (14)$$

and have solutions which are very important in many applications of science and engineering. Solutions to the Bessel equation are known known simply as Bessel functions. These solutions are known to originate by way of series expansion techniques for solving second order differential equations. Series solutions usually take a form similar to

$$y = \sum_{n=0}^{\infty} a_n x^n$$

were  $a_n$  are coefficients which will be computed by yet another formula known as a recurrence formula. Without going to deep lets assume that the solution we are search for has the form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n},$$

which will satisfy the recurrence formula.

Thus

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$$xy'(s, x) = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} \quad \text{and} \quad x^2 y''(s, x) = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n}$$

plugging these results into (14) we have

$$\sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + (x^2 - r^2) \sum_{n=0}^{\infty} a_n(s)x^{s+n} = 0$$

Again skipping much of the technical math, we can say that when  $r$  is not a negative integer our solution to the Bessel equation is

$$y(r, x) = x^r \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(n!)(r+1)\dots(r+n)} \left(\frac{x}{2}\right)^{2n} \quad (15)$$

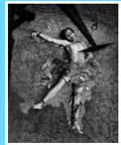
The formula for the coefficient in (15) is quit an ugly beast. However, there is known function that comes up naturally called the gamma function which will help. The Gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x > 0.$$

What is important to us about this function is that

$$\Gamma(x+1) = x!,$$

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which greatly reduces the complexity of the coefficients in (15) allowing us to denote the Bessel equation as

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(r+n)!} \left(\frac{x}{2}\right)^{2n}$$

which is known as the **Bessel function of the first kind**, of order  $r$ .

It can also be shown that there exists another solution  $Y_r$  known as the Bessel function of the second kind of order  $r$ .  $Y_r$  is linearly independent of  $J_r$ , thus the general solution to our Bessel equation of order  $r$  is a linear combination of the two functions, as follows

$$y_r(x) = \alpha J_r(x) + \beta Y_r(x) \quad x > 0 \quad (16)$$

This is our solution to equation (14) where  $\alpha$  and  $\beta$  are constants determined from our initial conditions

There are a few properties of  $y_r(x)$  that are important in light of our analysis of the pendulum thus far. Keep in mind the behavior we have seen as we note that,

- The extremes of  $|y_r(x)|$  decay monotonically to zero as  $x$  goes to infinity
- Though we will not prove it,  $y_r(x)$  oscillates like a decaying sinusoid about  $y = 0$ .

In order to further analyze the behavior of the pendulum we need to determine  $\alpha$  and  $\beta$ . The derivative of  $J_r$  is

$$J'_r = J_{r-1} - \left(\frac{r}{x}\right) J_r,$$

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and likewise that the derivative of  $Y_r$  is

$$Y_r' = Y_{r-1} - \left(\frac{r}{x}\right) Y_r.$$

Therefore, the derivative of our general solution (16) is

$$\alpha \left( J_0 - \left(\frac{1}{x}\right) J_1 \right) + \beta \left( Y_0 - \left(\frac{1}{x}\right) Y_1 \right). \quad (17)$$

Setting equation (16) to  $y_0$  and equation (17) to  $y_0'$  we can solve for  $\alpha$  and  $\beta$ . This is not an easy task, and it is made substantially simpler through the use of a numerical solver like MATLAB. Once  $\alpha$  and  $\beta$  are found the general solution can be solved and  $x$  and  $y$  computed. Using equations (12) and (13) we can then find  $\theta$  and  $t$ . Thus we now have an analytical method for determining the behavior of  $\theta$  over time. Of course, being that Bessel functions are infinite series, solving them by hand is not the most time efficient of activities. Therefore, we will again use MATLAB to plot our Bessel determined  $\theta$  vs. time using  $a = 1$  m,  $b = 0.1$  m/s,  $g = 9.8$  m/s<sup>2</sup> (normal gravity on earth),  $\theta_0 = \pi/20$  (we use a small  $\theta$  in accordance with our approximation  $\sin(\theta) = \theta$  for small  $\theta$ ), and  $\theta_0' = 0$  (we start the pendulum at rest), which gives us an  $\alpha \approx -1.2918$  and  $\beta \approx 0.8713$ . The plot of which can be seen below and is our familiar decaying angle.

## 8. Conclusion

We have now modeled the lengthening pendulum using two different methods. Perhaps now we should compare the results of these models to Poe's descriptions. As stated earlier, Poe described a pendulum that starts with a sweep that



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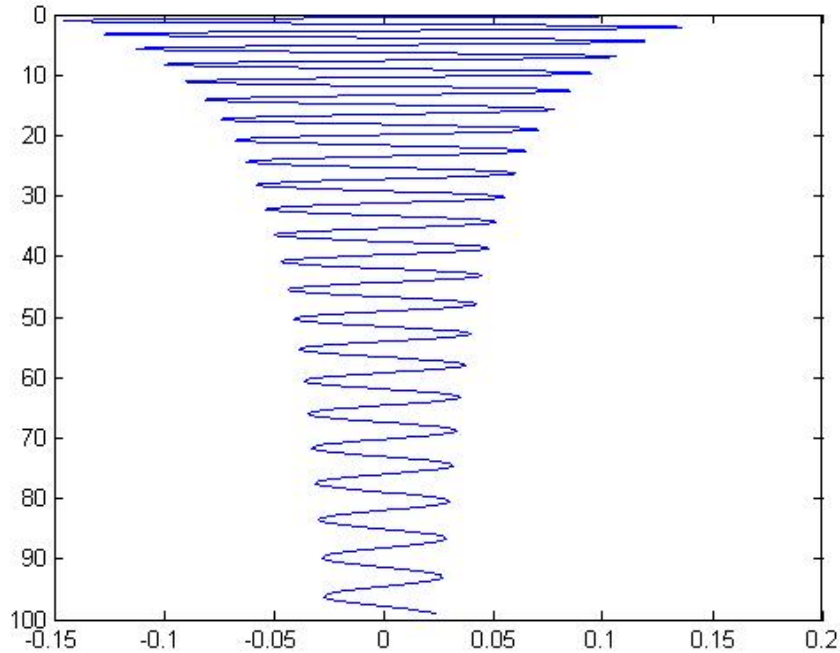


Figure 5: The plot of the Bessel Function transformed back to  $\theta$  and  $t$

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is “brief, and [...]slow” but, which by the time it reaches its victim, possess a “terrifically wide sweep (some thirty feet or more) and the hissing vigour of its descent, sufficient to sunder these very walls of iron” and which swings through an arc so great that “the vibration of the pendulum was at right angles to [the victims] length” implying that the pendulum increases not only in sweep, but in its angle of motion, and in its speed. However, as we have seen, in real life such a pendulum has a decaying angle of motion and a decaying velocity, both angular and linear. And, although the sweep of such a pendulum does increase, even if the pendulum is started at the maximum possible angle  $\pi/2$  (which is horizontal in our model and would mean that the pendulum started being pressed against the ceiling of the chamber) and allowed to descend for 98.8 seconds with the cable lengthening at 0.1 m/s with an initial length of 1 m, when it reaches its maximum length of  $\approx 11$  m, it will have a sweep of slightly less than 6 ft. which is not a “terrifically wide sweep (some thirty feet or more)”. When the pendulum reaches the victims chest it would have a velocity of  $\approx 2.5$  m/s , one could imagine it merely stopping on the intended victims chest rather than ripping through it, which would make Poe’s pendulum, in reality, a less than terrifying torture device, as the victim, rather than being driven mad by the “rushing oscillations of the steel”, would be relieved to see the blade slowing down.

## 9. Acknowledgements

Special thanks to Professor Dave Arnold for his infinite knowledge of all things, especially L<sup>A</sup>T<sub>E</sub>X and differential, equations and Shawn Rutan for being so easily

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distracted and providing invaluable assistance in time of need.

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Edgar Allan Poe's

THE PIT

AND THE

PENDULUM

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## 10. Appendix

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The following contains the MATLAB commands used to generate the images used in this paper.

This is the ODE45 code.

```
function project_anglefunction
close all
clc

% using ODE45 to solve for theta
ops=odeset('RelTol',1e-10);
a=1; b=.1; theta_0=pi/2; thetap_0=0; g=9.8; tinit=0; tfinal=98.8
[t,u]=ode45(@exactanswer,[tinit,tfinal],[theta_0,thetap_0],ops,a,b);

% angular displacement
figure
plot(u(:,1),t)
set(gca,'ydir','reverse')
title('Decaying angle')

% the path of the pendulum
figure
plot((a+b*t).*sin(u(:,1)),(a+b*t).*cos(u(:,1)))
```



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```
set(gca,'ydir','reverse')
% title('The Path of the Pendulum')
```

```
% Sweep
figure
plot((a+b.*t).*(u(:,1)),t)
set(gca,'ydir','reverse')
title('Increasing Sweep')
```

```
% angular velocity
figure
plot(u(:,2),t)
set(gca,'ydir','reverse')
title('Angular Velocity')
```

```
% linear velocity
figure
plot((a+b.*t).*u(:,2),t)
set(gca,'ydir','reverse')
title('Linear Velocity')
```

```
% curvilinear Velocity
figure
plot((a+b.*t).*u(:,2) + u(:,1)*b,t)
set(gca,'ydir','reverse')
title('Decaying Curvilinear Velocity')
```

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```

%maxima of theta
[W,R]=size(u(:,1));
dec=[];
dect=[];
for count = 1:W-2
    if (u(count+1,1)>u(count,1)) & (u(count+1,1)>u(count+2,1))
        dec=[dec,u(count+1,1)];
        dect=[dect,t(count+1,1)];
    end
end
end

```

```

% The plot of the decay of theta
figure
plot(dect,dec,'r')

```

```

% The plot of the sweep
hold on
plot(dect,(a+b.*dect).*dec,'b')

```

```

%maxima of theta'
[W,R]=size(u(:,2));
decpr=[];
decpt=[];
for count = 1:W-2

```

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```

if (u(count+1,2)>u(count,2)) & (u(count+1,2)>u(count+2,2))
    decp=[decp,u(count+1,2)];
    decpt=[decpt,t(count+1)];
end
end

% angular velocity
plot(decpt,decp,'g')

% linear velocity
plot(decpt,(a+b.*decpt).*decp,'k')
legend('angle','sweep','angular velocity','linear velocity')
title('The Behavior of the Maxima')

% The velocity of the pendulum
question=(a+b*t).*u(:,2)

function uprime=exactanswer(t,u,a,b)
uprime=zeros(2,1);
uprime(1)=u(2);
uprime(2)=-2*b*u(2)./(a+b.*t)-9.8*sin(u(1))./(a+b.*t);

```

This is the Bessel code.

```

function project_anglefunction
close all
clc

```

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```

% initial conditions and variables
ops=odeset('RelTol',1e-10);
a=1; b=.1; theta_0=pi/20; thetap_0=0; g=9.8; tinit=0; tfinal=99;

% initial conditions for the function of x & y
xinit=(2/b)*sqrt((a+b*tinit)*g); xfinal=(2/b)*sqrt((a+b*tfinal)*g);
y_0=sqrt(a+b*tinit)*theta_0;
yp_0=thetap_0*(a+b*tinit)/sqrt(g)+(theta_0*b)/(2*sqrt(g));

% solving the function of x & y
[xb,yb]=ode45(@answerxy,[xinit,xfinal],[y_0,yp_0],ops,a,b);

% The function of x & y recomputed back to theta & t
figure
tb=((xb(:,1).^2*b)/(4*g))-(a/b);
plot(yb(:,1)./sqrt(a+b.*tb),tb)
set(gca,'ydir','reverse')
title('The function x & y recomputed to theta')
%axis([-0.15,.2,-50,0])

% the plot of x vs y
figure
plot(yb(:,1),xb)
set(gca,'ydir','reverse')
title('the plot of y vs. -x')

```

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```
% solving the approximate equation  $L\theta'' + 2L'\theta' + g\theta = 0$ 
[tappr,uappr]=ode45(@apprtheta,[tinit,tfinal],...
    [theta_0,thetap_0],ops,a,b);
```

```
figure
plot(uappr(:,1),tappr)
set(gca,'ydir','reverse')
title('approximate equation  $L\theta'' + 2L'\theta' + g\theta = 0$ ')
```

```
% using matlab's automated bessell
syms a b g x alphas betas;
y=alphas*besselj(1,x)+betas*bessely(1,x)
yp=diff(y,x)
%x=2*sqrt(a*g)/b
x=xinit;
y=subs(y);
eq1=y-y_0; %pi*sqrt(a)/20
yp=subs(yp);
eq2=yp-yp_0; %pi*b/(40*sqrt(g))
[alphas,betas]=solve(eq1,eq2,alphas,betas);
a=1;b=.1;g=9.8;
betas=subs(betas)
alphas=subs(alphas)
x=linspace(xinit,xfinal,2000);
z=alphas*besselj(1,x)+betas*bessely(1,x);
figure
```

“Down and still down it came – to cross the region of the heart!...” POE



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```

plot(z,x)
set(gca,'ydir','reverse')
title('auto_bessel')

% plotting matlabs automated_bessel back to t and theta
figure
tbess=((x.^2*b)/(4*g)-a/b);
thetabess=z./sqrt(a+b.*tbess);
plot(thetabess,tbess)
set(gca,'ydir','reverse')
title('MATLabs Automated Bessel converted back to t and theta')

```

```

function uprimea=apprtheta(t,ua,a,b)
uprimea=zeros(2,1);
uprimea(1)=ua(2);
uprimea(2)=-2*b*ua(2)./(a+b.*t)-9.8*ua(1)./(a+b.*t);

```

```

function uprimeb=answerxy(x,ub,a,b)
uprimeb=zeros(2,1);
uprimeb(1)=ub(2);
uprimeb(2)=-ub(2)/x-ub(1)+ub(1)/x^2;

```

This is the code for an animation of the pendulum's motion.

```

function project_anglefunction
close all
clc

```

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```

% using ODE45 to solve for theta
ops=odeset('RelTol',1e-10);
a=.1; b=.1; theta_0=pi/2; thetap_0=0; g=9.8; tinit=0; tfinal=50;
increment=500;
test=linspace(tinit,tfinal,increment);
[t,u]=ode45(@anim,test,[theta_0,thetap_0],ops,a,b);

% the animation of the pendulum
parabstart=-.5;
parabstop=.5;
tparab=linspace(parabstart,parabstop);

tback=[];
uback=[];
figure

for time = 1:increment
    plot((a+b*t).*sin(u(:,1)),(a+b*t).*cos(u(:,1)),'k')
    hold on
    tback=[tback,t(time)];
    uback=[uback,u(time,1)];
    plot((a+b*tback).*sin(uback),(a+b*tback).*cos(uback),'g')
    plot(-tparab+(a+b*t(time)).*sin(u(time,1)),...
        -tparab.^2+(a+b*t(time)).*cos(u(time,1)),'r')
    plot([0,(a+b*t(time)).*sin(u(time,1))],...

```

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```

        [0,(a+b*t(time)).*cos(u(time,1))])
    set(gca,'ydir','reverse')
    hold off
    axis([-sin(max(u(:,1)))+2*parabstart,...
        sin(max(u(:,1)))+2*parabstop,0,(a+b*tfinal)])
    M(time) = getframe;
end

% play movie
%movie(M,0)

function uprime=anim(t,u,a,b)
uprime=zeros(2,1);
uprime(1)=u(2);
uprime(2)=-2*b*u(2)./(a+b.*t)-9.8*sin(u(1))./(a+b.*t);

```

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