

College of the Redwoods

Math 55: Ordinary Differential Equations



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The Motion of Pumping On A Swing

Johnathon W. Jackson



Pumping On A Swing

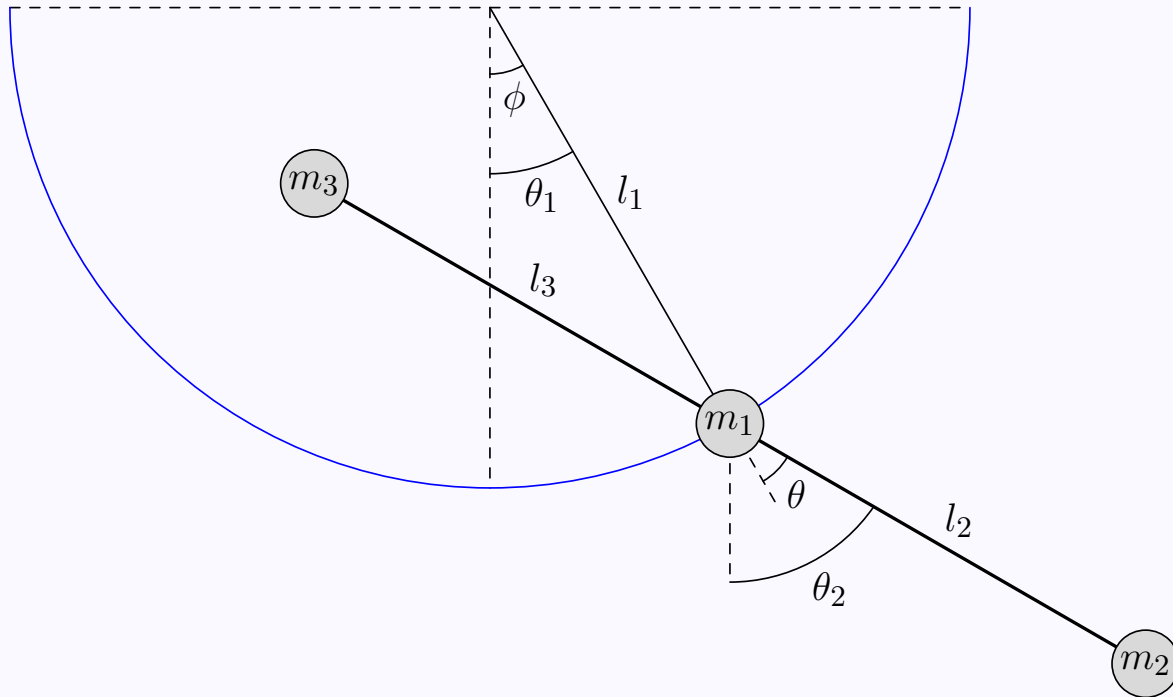


Layout

- Kinetic energy
- Potential energy
- Lagrangian
- Euler's formula
- Approximation of the terms
- The harmonic system
- The parametric system
- Linear and exponential resonance



The System



Variable Dictionary

- ϕ is the angle in which the system rotates about the origin.
- θ is the angle in which the masses m_2 and m_3 rotates about the center mass (m_1).
- θ_1 is equal to the angle ϕ .
- θ_2 is equal to the sum of the angles $\phi + \theta$.



Energy

- K = Total kinetic energy
- U = Total potential energy
- $L = K - U$



Position

Coordinates of mass m_1 .

$$x_1 = l_1 \sin(\theta_1)$$

$$y_1 = l_1 \cos(\theta_1)$$

Coordinates of mass m_2 .

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$

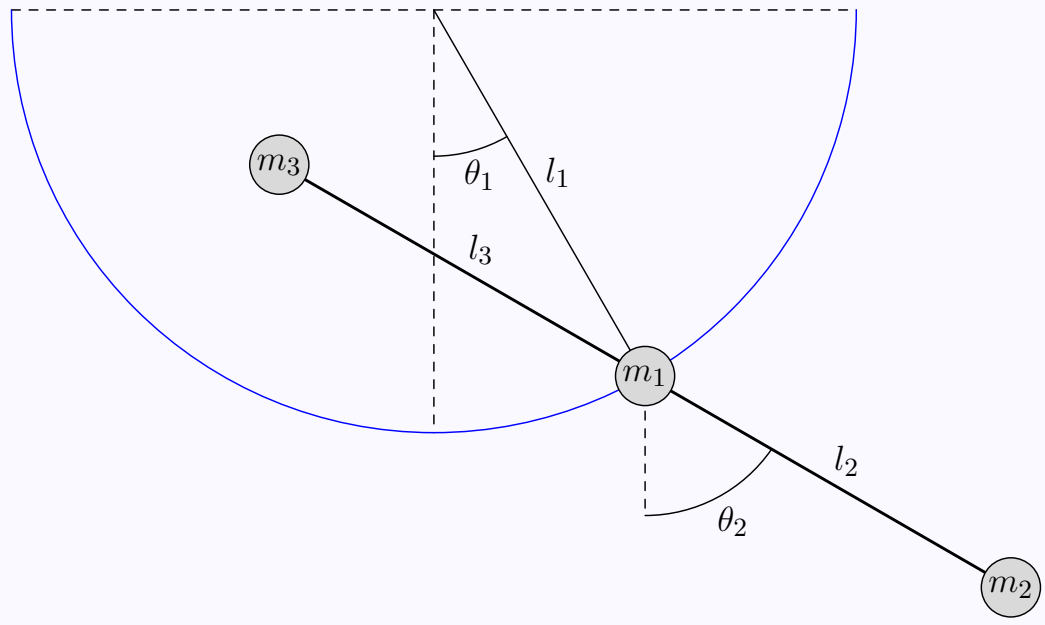
$$y_2 = l_1 \cos(\theta_1) + l_2 \cos(\theta_2)$$

Coordinates of mass m_3 .

$$x_3 = l_1 \sin(\theta_1) - l_3 \sin(\theta_2)$$

$$y_3 = l_1 \cos(\theta_1) - l_3 \cos(\theta_2)$$





coordinates :

$$(x_1, y_1) = (l_1 \sin(\theta_1), l_1 \cos(\theta_1))$$
$$(x_2, y_2) = (l_1 \sin(\theta_1) + l_2 \sin(\theta_2), l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$$
$$(x_3, y_3) = (l_1 \sin(\theta_1) - l_3 \sin(\theta_2), l_1 \cos(\theta_1) - l_3 \cos(\theta_2))$$



Velocity

Velocity of mass m_1 .

$$\dot{x}_1 = l_1 \cos(\theta_1) \dot{\theta}_1$$

$$\dot{y}_1 = -l_1 \sin(\theta_1) \dot{\theta}_1$$

Velocity of mass m_2 .

$$\dot{x}_2 = l_1 \cos(\theta_1) \dot{\theta}_1 + l_2 \cos(\theta_2) \dot{\theta}_2$$

$$\dot{y}_2 = -l_1 \sin(\theta_1) \dot{\theta}_1 - l_2 \sin(\theta_2) \dot{\theta}_2$$

Velocity of mass m_3 .

$$\dot{x}_3 = l_1 \cos(\theta_1) \dot{\theta}_1 - l_3 \cos(\theta_2) \dot{\theta}_2$$

$$\dot{y}_3 = -l_1 \sin(\theta_1) \dot{\theta}_1 + l_3 \sin(\theta_2) \dot{\theta}_2.$$



Velocity Squared

$$v_1^2 = (\dot{x}_1^2 + \dot{y}_1^2)$$

$$v_2^2 = (\dot{x}_2^2 + \dot{y}_2^2)$$

$$v_3^2 = (\dot{x}_3^2 + \dot{y}_3^2)$$

Substituting in the computed values of the \dot{x}_i and \dot{y}_i we have

$$v_1^2 = l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 + l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2$$

$$v_2^2 = (l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 + 2l_1 l_2 \cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \cos(\theta_2)^2 \dot{\theta}_2^2) \\ + (l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2 + 2l_1 l_2 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \sin(\theta_2)^2 \dot{\theta}_2^2)$$

$$v_3^2 = (l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 - 2l_1 l_3 \cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_3^2 \cos(\theta_2)^2 \dot{\theta}_2^2) \\ + (l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2 - 2l_1 l_3 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_3^2 \sin(\theta_2)^2 \dot{\theta}_2^2)$$



Kinetic Energy

The equation for the kinetic energy is given by the equation

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$$

Substituting the equations from the previous page in for the squared velocities we get

$$K = \frac{1}{2}l_1^2\dot{\theta}_1^2[m_1 + m_2 + m_3] + \frac{1}{2}\dot{\theta}_2^2[m_2l_2^2 + m_3l_3^2] \\ + l_1\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)][m_2l_2 - m_3l_3].$$



Potential Energy

The equation for the potential energy is given by the equation

$$U = -m_1gy_1 - m_2gy_2 - m_3gy_3$$

Substituting the equations for the y -coordinates in for y we get

$$U = -[gl_1 \cos(\theta_1)(m_1 + m_2 + m_3)] - [g \cos(\theta_2)(m_2l_2 - m_3l_3)]$$



Substitutions

We eliminate a number of parameters for simplicity. We let

$$M = m_1 + m_2 + m_3$$

$$I_1 = Ml_1^2$$

$$I_2 = m_2l_2 + m_3l_3$$

$$N = m_3l_3 - m_2l_2$$

$$\theta_1 = \phi$$

$$\theta_2 = \phi + \theta$$



Our Energy Equations Become

Kinetic energy:

$$K = \frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2(\dot{\phi} + \dot{\theta})^2 - l_1N\dot{\phi}(\dot{\phi} + \dot{\theta})\cos(\theta).$$

Potential energy:

$$U = -Mgl_1\cos(\phi) + Ng\cos(\phi + \theta).$$



Lagrangian

The Lagrangian for the swing system is

$$L = K - U$$

Substituting the values for the kinetic and potential energies in for K and U we get

$$L = \frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2(\dot{\phi} + \dot{\theta})^2 - l_1N\dot{\phi}(\dot{\phi} + \dot{\theta})\cos(\theta) \\ + Mgl_1\cos(\phi)(M) - Ng\cos(\phi + \theta)$$



Euler-Lagrange Equation

The Euler-Lagrange Equation is given by the equation

$$0 = \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\phi}} \right) - \left(\frac{\delta L}{\delta \phi} \right).$$

Finally, solving for $(d/dt)[(\delta L)/(\delta \dot{\phi})] - (\delta L)/(\delta \phi)$ we get

$$\begin{aligned} 0 = & I_1 \ddot{\phi} + I_2 \ddot{\phi} + I_2 \ddot{\theta} - 2l_1 N \cos(\theta) \ddot{\phi} - l_1 N \cos(\theta) \ddot{\theta} \\ & + 2l_1 N \sin(\theta) \dot{\phi} \dot{\theta} + l_1 N \sin(\theta) \dot{\theta}^2 + Mgl_1 \sin(\phi) \\ & - Ng(\sin(\phi) \cos(\theta) + \cos(\phi) \sin(\theta)) \end{aligned}$$



Approximations

Some of the approximation we have made include the following:

- approximation of sin and cosine factors using Taylor series.
- approximation of sin and cosine factors where $\sin(\phi) \cong 0$ and $\cos(\phi) \cong 1$ for small angles of ϕ .
- approximation of sin and cosine factors that are of some degree n to a sum of terms of degree 1 and eliminating terms with frequencies at some multiple of the natural frequency.



More Substitutions

We now make the following substitutions

$$I_0 = [(I_1 + I_2) + 2l_1N(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4)]$$

$$K_0 = [Mgl_1 + Ng(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4)]$$

$$\omega_0 = K_0/I_0$$

$$F = \theta_0[(\omega^2 I_2 + N[(g - l_1\omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4))]/I_0]$$

$$A = -\frac{1}{4}Ng(\theta_0^2 - \frac{1}{12}\theta_0^4)/I_0$$

$$B = l_1N\omega[\theta_0^2 - \frac{1}{12}\theta_0^4]/I_0$$

$$C = -\frac{1}{2}l_1N(\theta_0^2 - \frac{1}{12}\theta_0^4)/I_0.$$



The Approximated Swing System

The swing system becomes

$$\ddot{\phi} + \omega\phi \cong F \cos(\omega t) + A \cos(2\omega t)\phi + B \sin(2\omega t)\dot{\phi} + C \cos(2\omega t)\ddot{\phi}. \quad (1)$$

Notice that the F term isn't dependant on ϕ whereas the other three terms on the right hand side are ϕ -dependant. The F term is driving force and the other three terms are parametric terms, in that they are functions of the angle ϕ (they have a time-dependant piece).



The Forced Harmonic System

We can approximate the motion of the system when ϕ is small by looking at just F -term. Thus our new system becomes

$$\ddot{\phi} + \omega\phi \cong F \cos(\omega t).$$



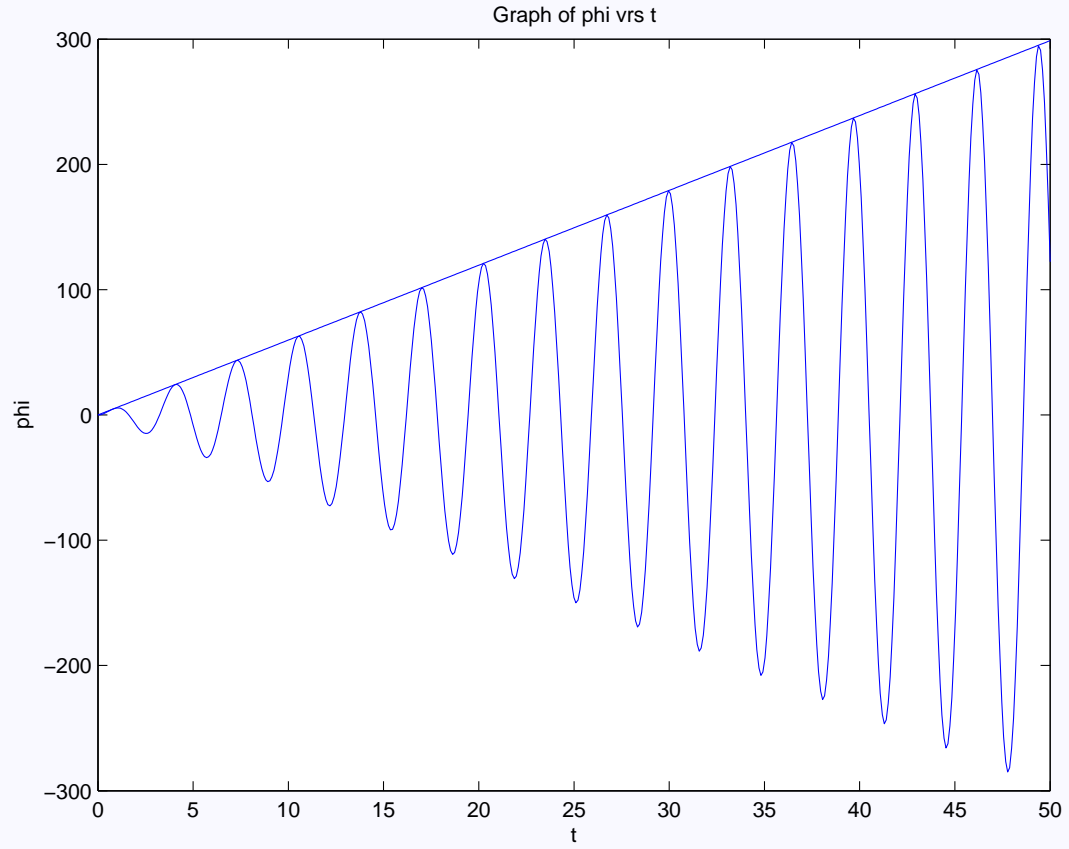
When solving the differential equation for ϕ , with initial conditions that the swing initially at rest at time $t = 0$ ($\phi(0) = 0$), we get

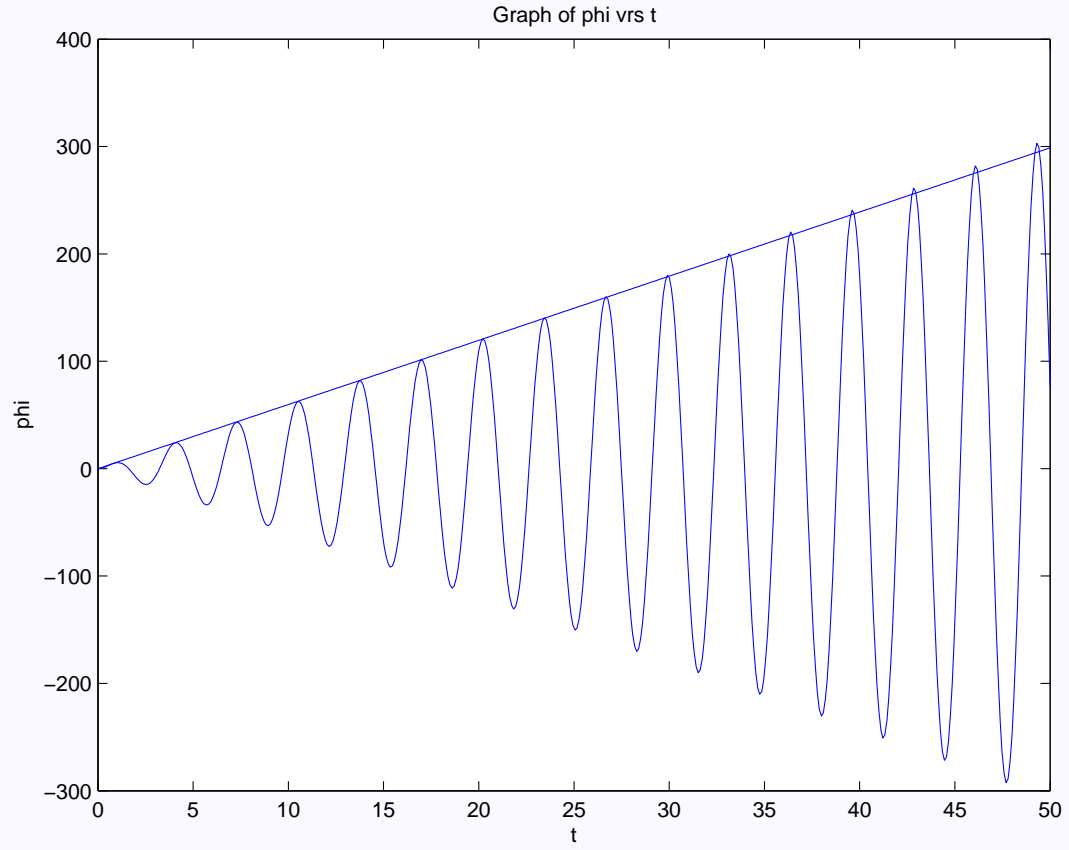
$$\phi \cong (Ft)/(2\omega_0) \sin(\omega_0 t),$$

which results in a linear growth per cycle

$$\Gamma_D = (F\pi)/(\omega_0^2).$$







Solving For The Parametric System

We can approximate the motion of the system when ϕ is large by looking at the ϕ -dependant terms. Thus our new system becomes

$$\ddot{\phi} + \omega\phi \cong A \cos(2\omega t)\phi + B \sin(2\omega t)\dot{\phi} + C \cos(2\omega t)\ddot{\phi}.$$





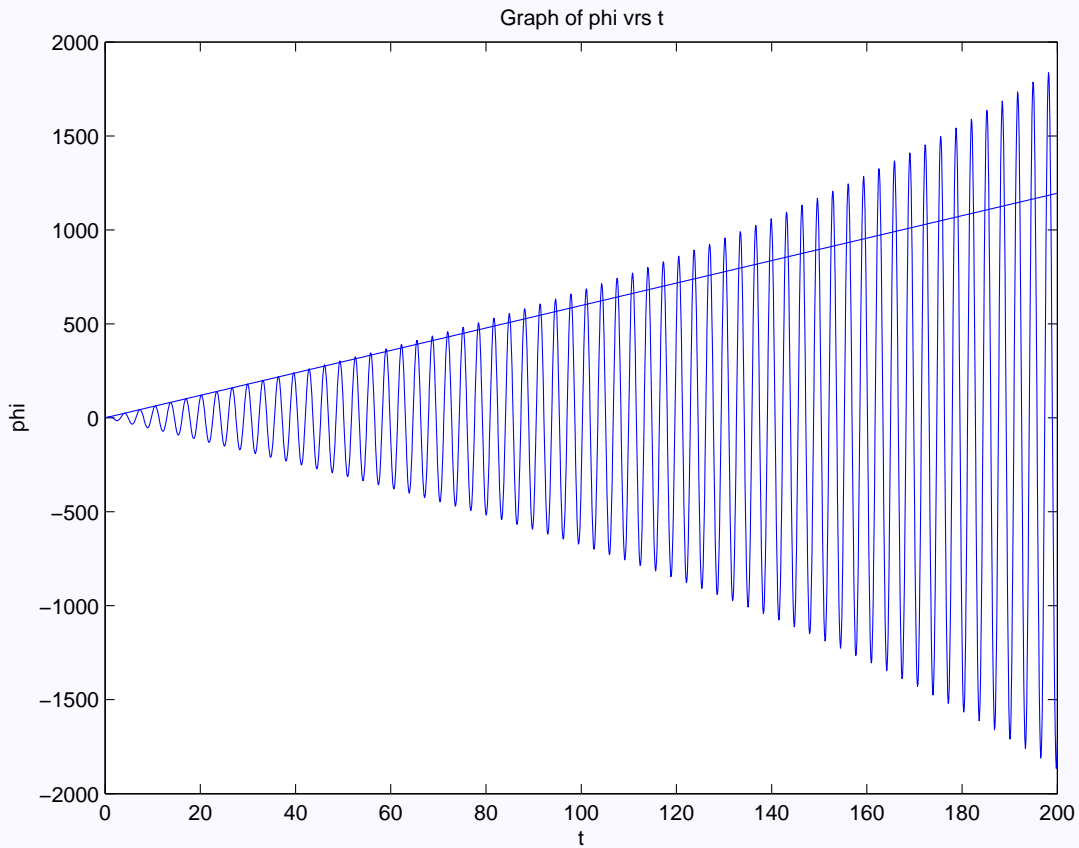
The difference between the forced harmonic oscillator system and the ϕ -dependant system is that an initial condition at time $t=0$ results in no amplitude growth in the system. Thus, we have to assume that the swing is pulled back to some initial angle of ϕ for there to be some positive contribution to the growth of the amplitude of the system. When solving the differential equation for ϕ , we get

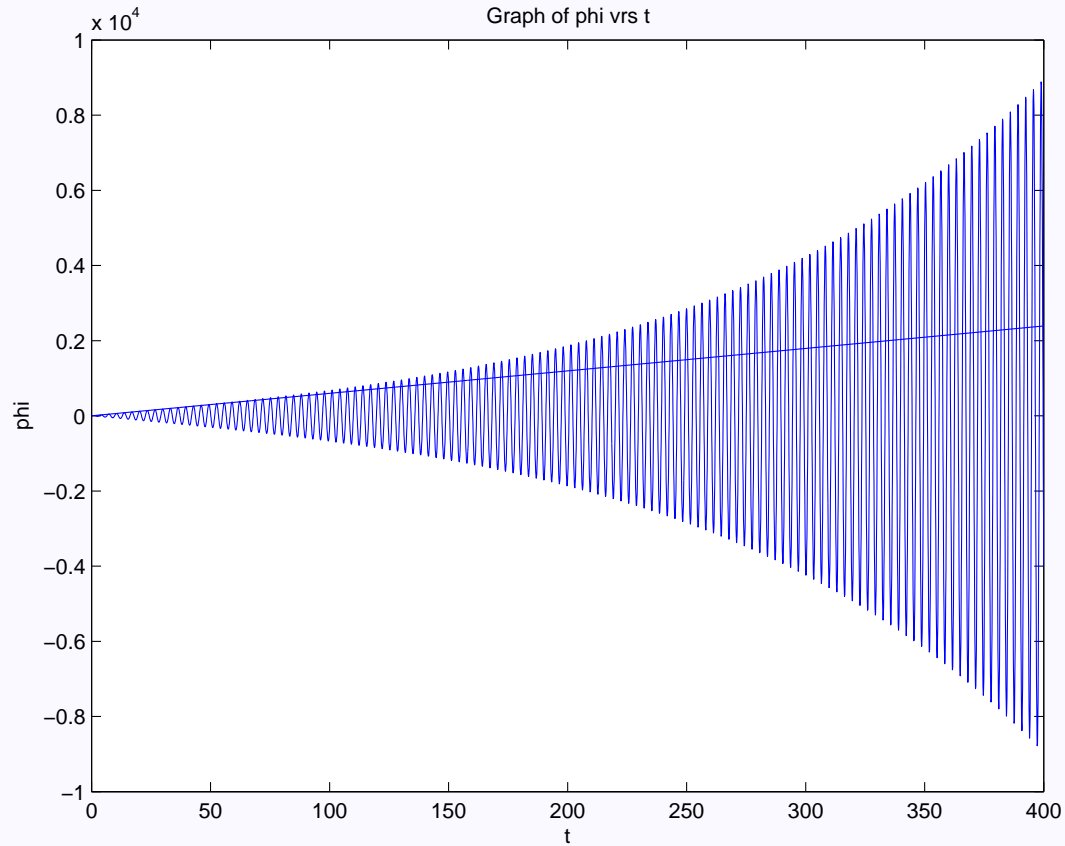
$$\phi \cong \pm \sqrt{2} |a| e^{-((A-\omega_0 B - \omega_0^2 C)t / (4\omega_0))} (\cos(\omega t) - \sin(\omega t))$$

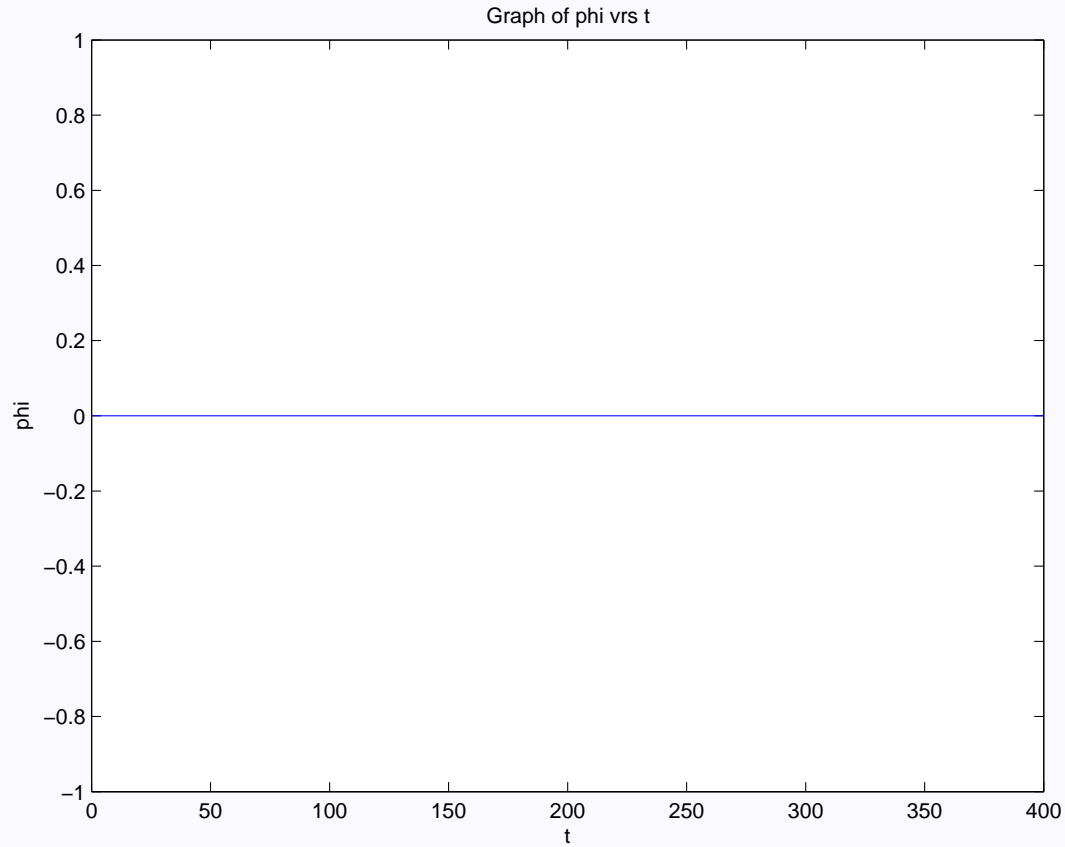
which results in an exponential growth per cycle

$$\Gamma_P = (2\pi\lambda) / (\omega_0)\phi_0.$$









Conclusion

By looking at the two systems, we can see that the forced harmonic system

$$\ddot{\phi} + \omega\phi \cong F \cos(\omega t)$$

appears to be the dominating term early in the system when ϕ is small. However, as ϕ approaches some critical angle

$$\phi_{critical} \cong \frac{8I_2}{3\theta_0 l_1 N}$$

when the growth per cycle of the forced harmonic system equals the growth of the parametric system

$$\ddot{\phi} + \omega\phi \cong A \cos(2\omega t)\phi + B \sin(2\omega t)\dot{\phi} + C \cos(2\omega t)\ddot{\phi},$$

the parametric terms start to dominate in the contribution of growth to the system.

