

Differential Equations
Spring 2003

A Differential Look at the Watt's Governor

by Tim Honn & Seth Stone

College of the Redwoods
Eureka, CA
Math dept.

email: timhonn@cox.net

email: lamentofseth@hotmail.com



1/25



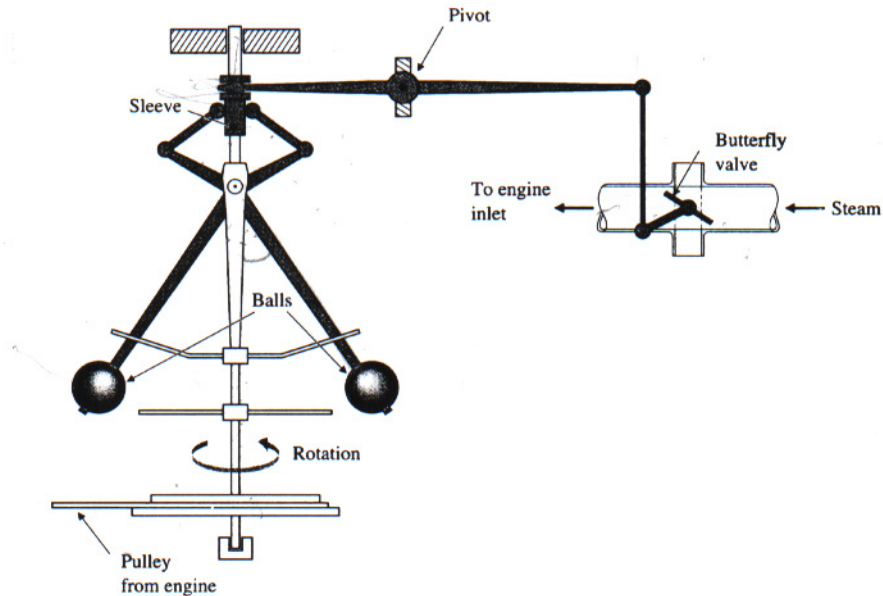
Introduction

Invented by James Watt in the late 1700s, a governor is an automated speed control that ushered in the industrial revolution.

- Mathematical model.
- Bifurcation.
- Damping.



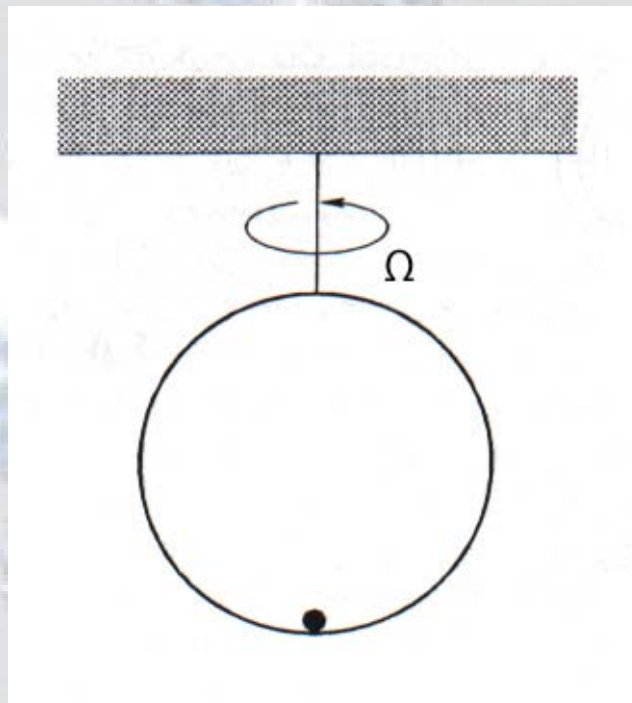
Watt's Governor



The Watt's governor controlling a steam engine.

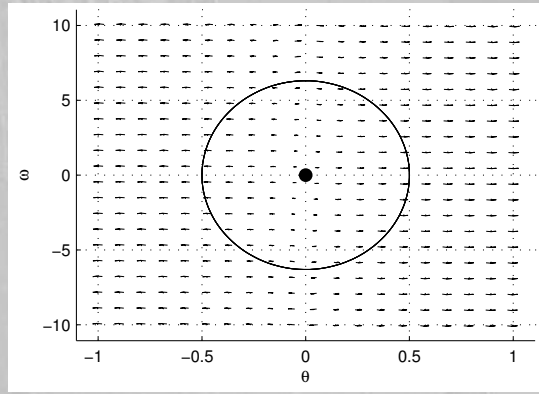


A Simplified Version

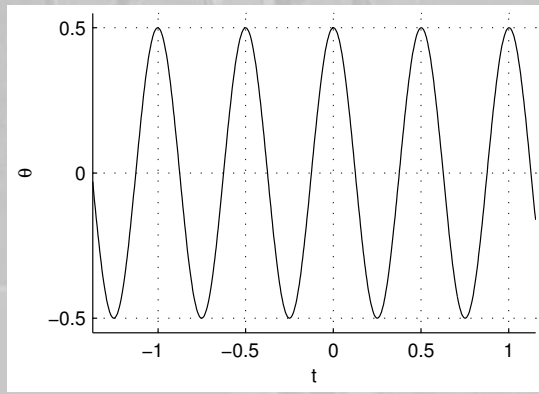


Ball-bearing in a rotating hoop.



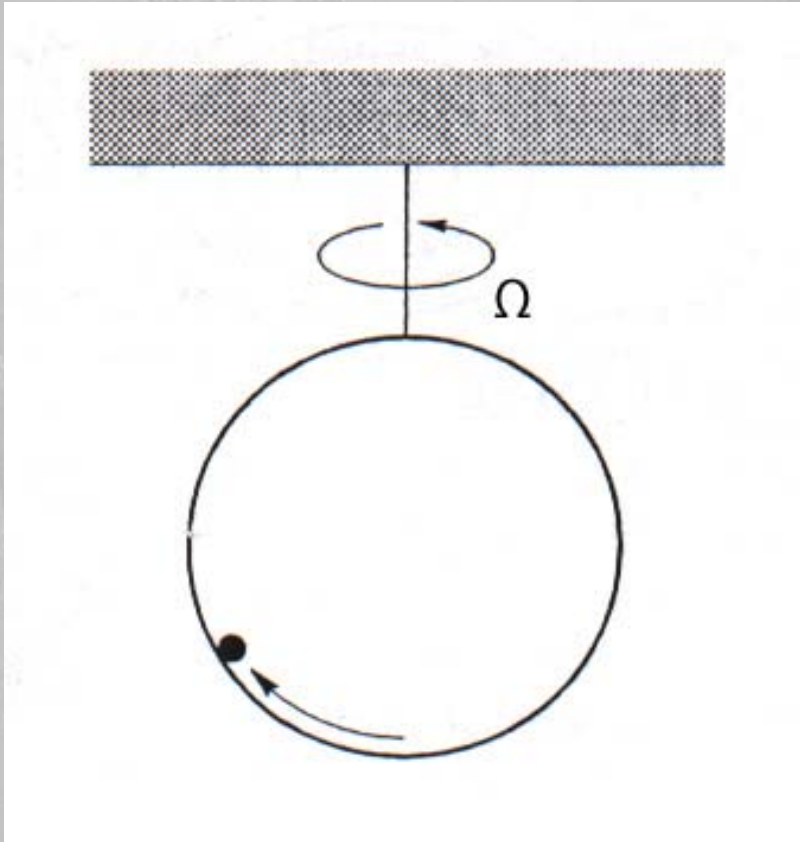


Phase plane for $\Omega = 1$ rad/sec.



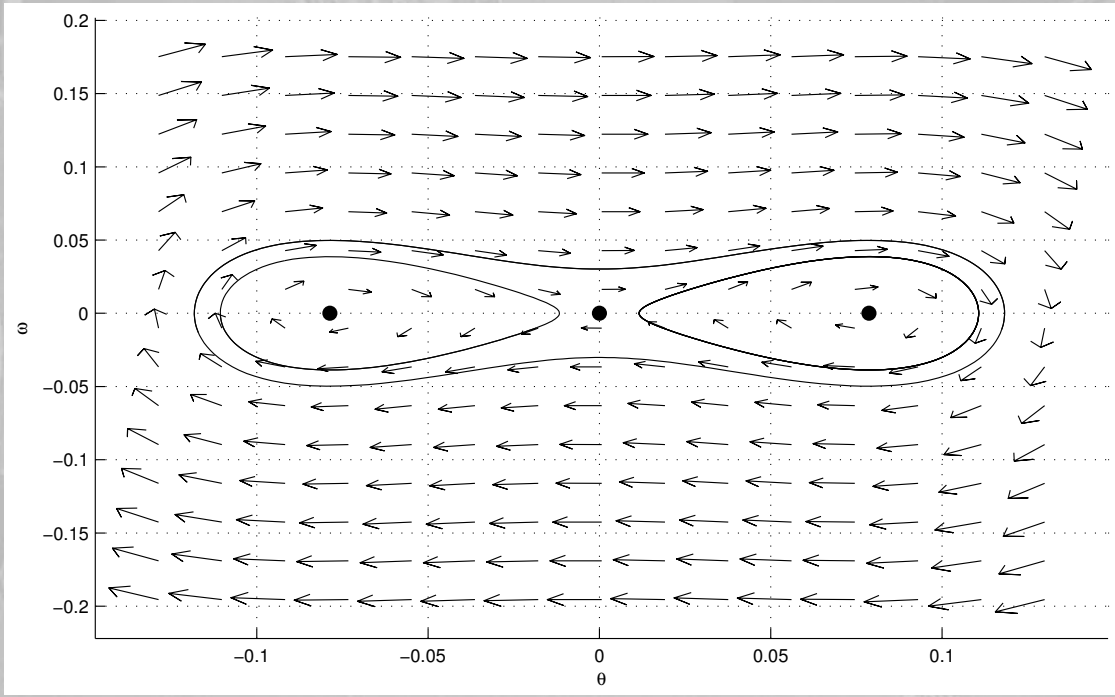
θ vs. t for $\Omega = 1$ rad/sec.





For $\Omega > 12 \text{ rad/sec}$ the ball moves towards a new equilibrium point.





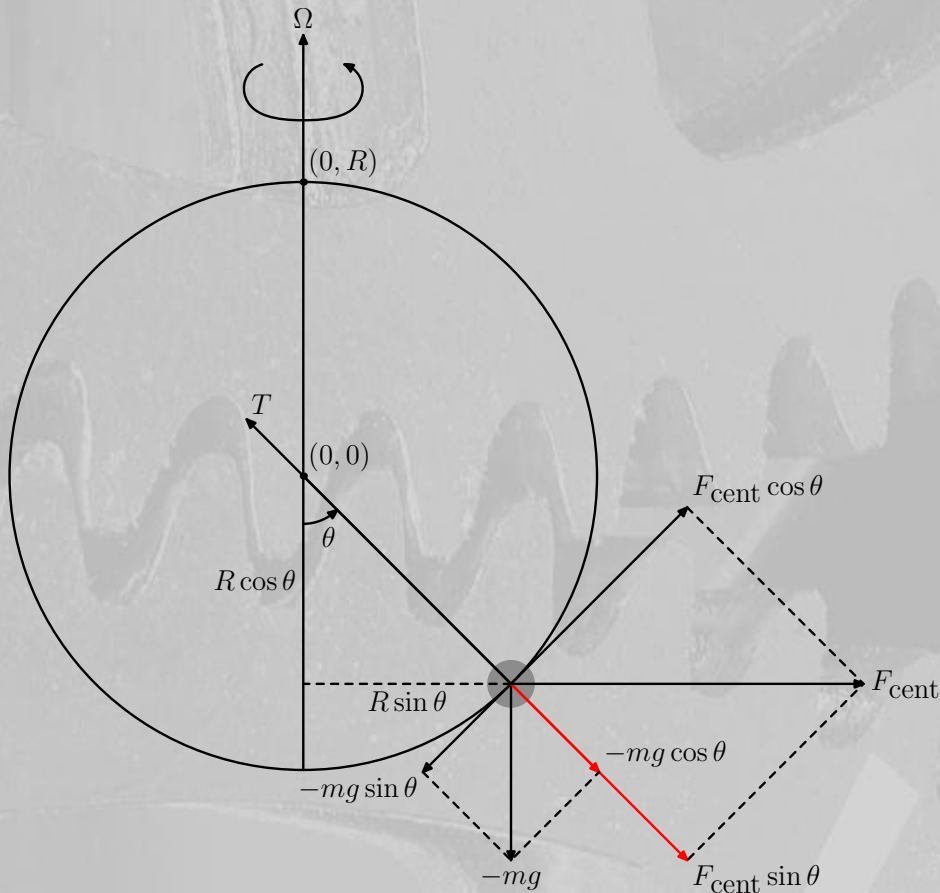
θ vs. t for $\Omega = 13$ rad/sec.



Identifying the Forces

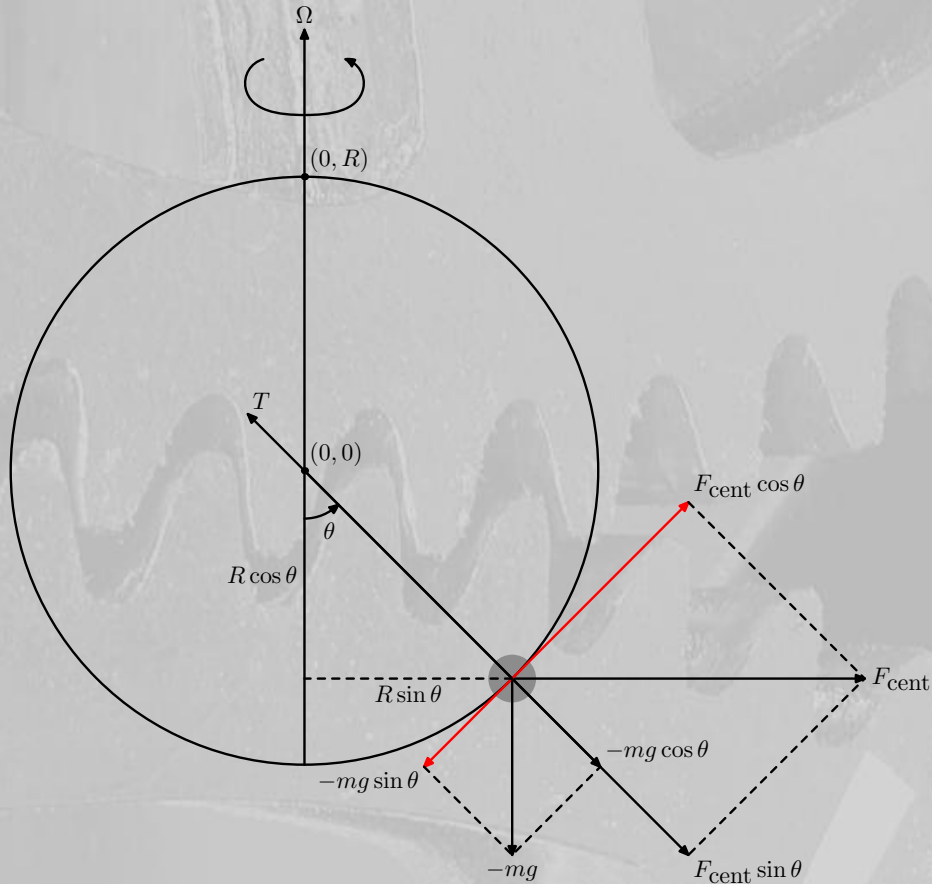
- Identify the forces that always balance.
- Identify the forces that do not always balance.
- Sum the forces to derive the equations.





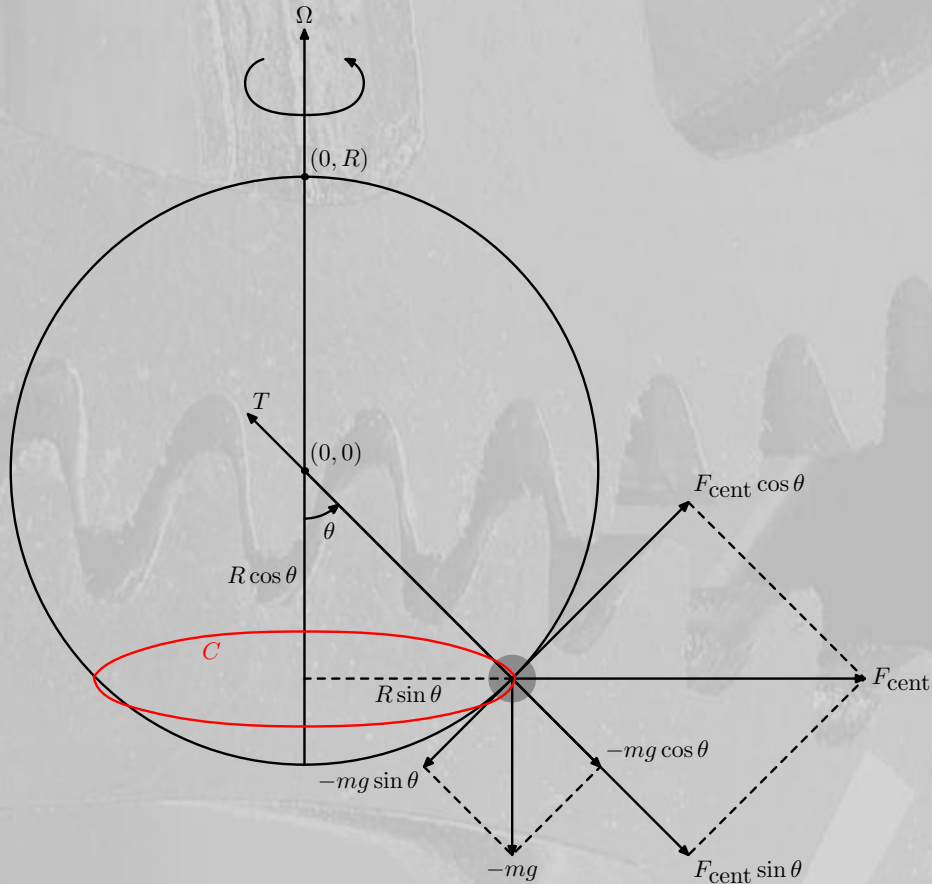
Forces opposing the normal force.





Tangential forces in the vertical plane.





The horizontal path of the ball.





Finding F_{cent}

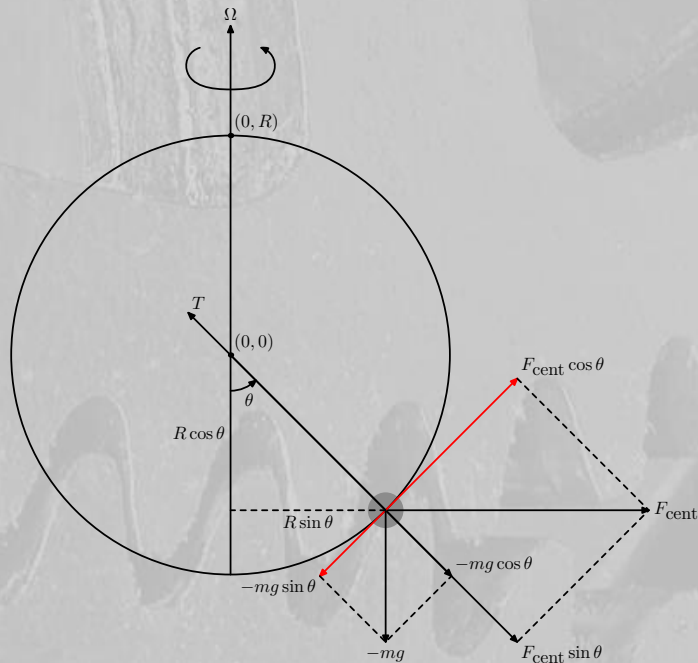
Recall the kinematic identities, and our values.

$$\begin{cases} v_{\text{lin}} = r v_{\text{ang}} \\ a_r = \frac{v_{\text{lin}}^2}{r} \\ F_{\text{cent}} = m a_r \end{cases} \quad \begin{cases} v_{\text{lin}} = (R \sin \theta) \Omega \\ a_r = \frac{[(R \sin \theta) \Omega]^2}{R \sin \theta} = (R \sin \theta) \Omega^2 \\ F_{\text{cent}} = m (R \sin \theta) \Omega^2 \end{cases}$$

In our case, Ω is the angular velocity v_{ang} , about the center of C and the radius is $R \sin \theta$. The centrifugal force acting on the ball is the mass times a_r .

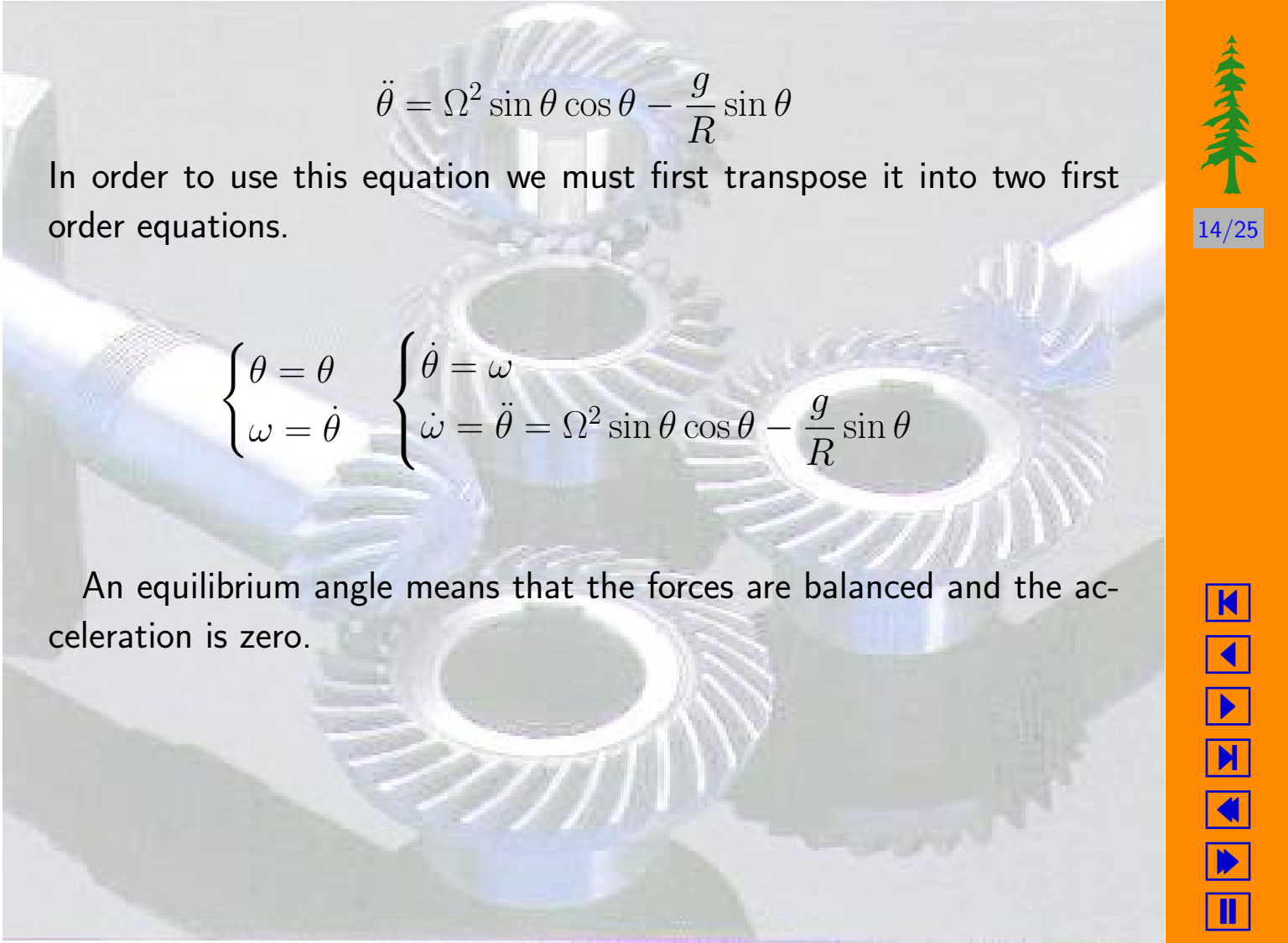
$$F_{\text{cent}} = m \Omega^2 R \sin \theta.$$





$$\begin{aligned} ma_T &= F_{\text{cent}} \cos \theta - mg \sin \theta \\ mR\ddot{\theta} &= m\Omega^2 R \sin \theta \cos \theta - mg \sin \theta \\ mR\ddot{\theta} &= m\Omega^2 R \sin \theta \cos \theta - mg \sin \theta \\ \ddot{\theta} &= \Omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta \end{aligned} \quad (1)$$




$$\ddot{\theta} = \Omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta$$

In order to use this equation we must first transpose it into two first order equations.

$$\begin{cases} \theta = \theta \\ \omega = \dot{\theta} \end{cases} \quad \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \ddot{\theta} = \Omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta \end{cases}$$

An equilibrium angle means that the forces are balanced and the acceleration is zero.



Set the right side equal to zero.

$$\ddot{\theta} = 0$$

$$\Omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta = 0$$

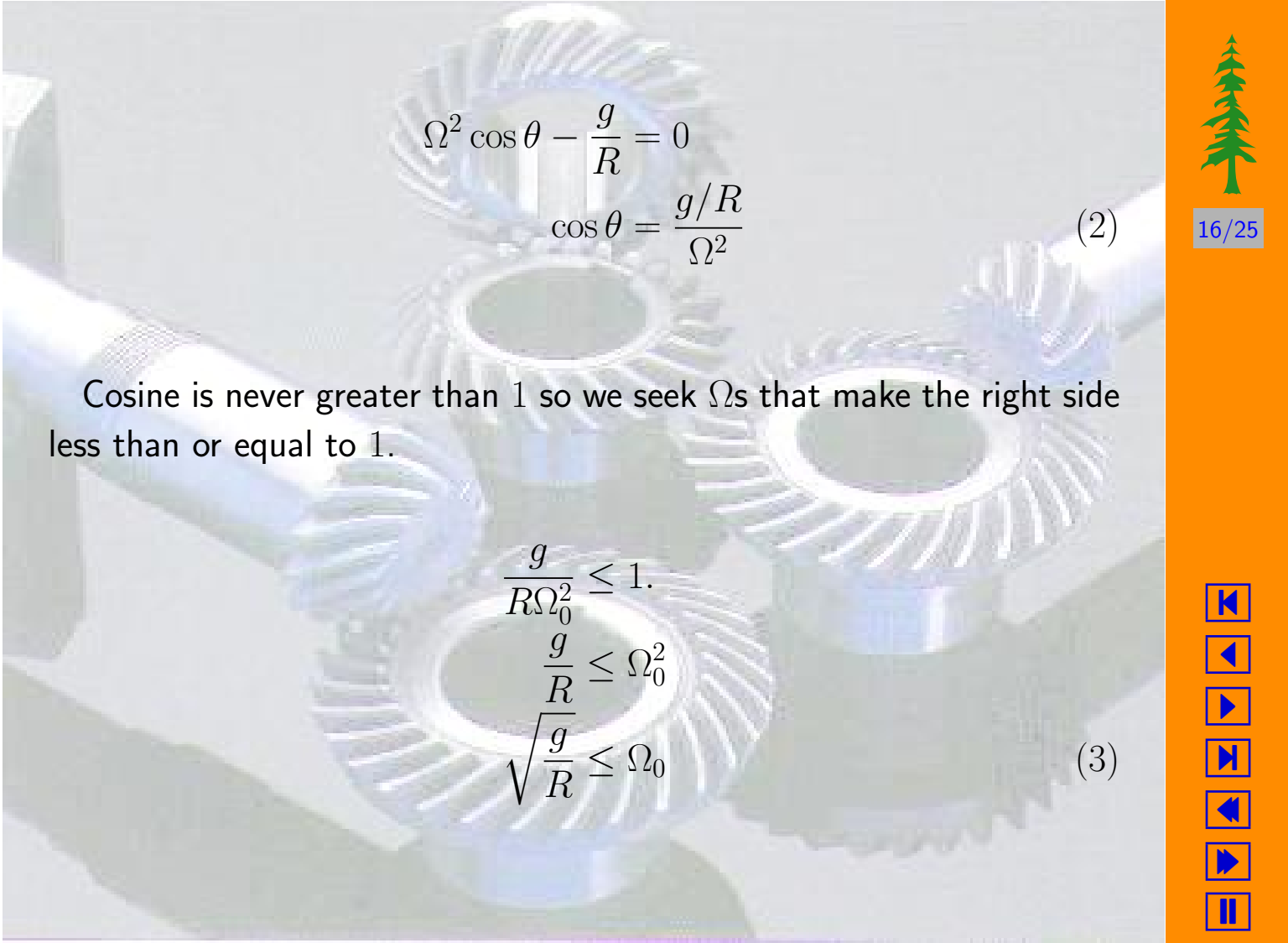
$$\sin \theta \left(\Omega^2 \cos \theta - \frac{g}{R} \right) = 0$$

Therefore,

$$\sin \theta = 0 \quad \text{or} \quad \Omega^2 \cos \theta - \frac{g}{R} = 0.$$

When $\sin \theta = 0$, $\theta = 0$ or π . To find other equilibrium angles we set the other factor equal to zero.




$$\begin{aligned}\Omega^2 \cos \theta - \frac{g}{R} &= 0 \\ \cos \theta &= \frac{g/R}{\Omega^2}\end{aligned}\quad (2)$$

Cosine is never greater than 1 so we seek Ω s that make the right side less than or equal to 1.

$$\begin{aligned}\frac{g}{R\Omega_0^2} &\leq 1. \\ \frac{g}{R} &\leq \Omega_0^2 \\ \sqrt{\frac{g}{R}} &\leq \Omega_0\end{aligned}\quad (3)$$



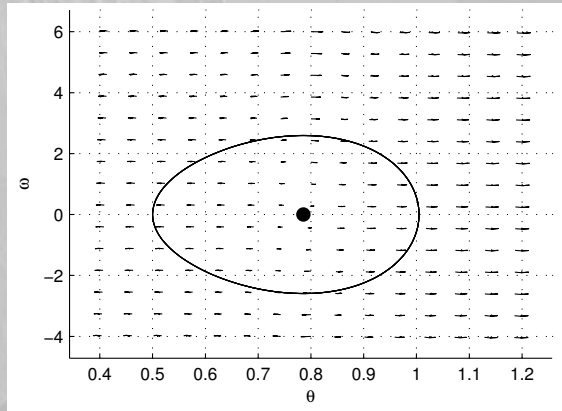
In our case the Ω where bifurcation occurs is,

$$\begin{aligned}\sqrt{\frac{9.8}{.06}} &\leq \Omega_0 \\ 12.78 &\leq \Omega_0.\end{aligned}\tag{4}$$

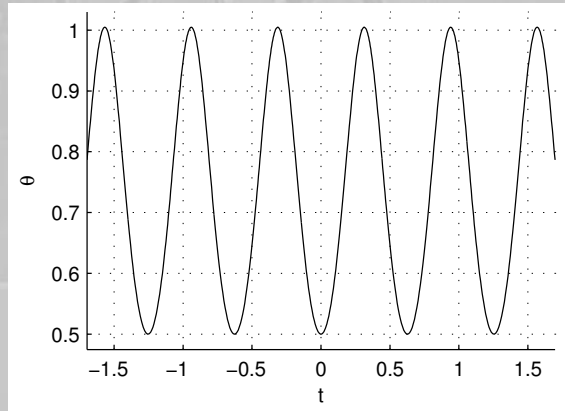
Now we find the Ω that produces $\theta = \pi/4$.

$$\begin{aligned}\cos \theta &= \frac{g/R}{\Omega^2} \\ \cos \frac{\pi}{4} &= \frac{9.8/.06}{\Omega^2} \\ \sqrt{\frac{9.8}{.06 \cos \frac{\pi}{4}}} &= \Omega \\ 15.2 &= \Omega.\end{aligned}\tag{5}$$





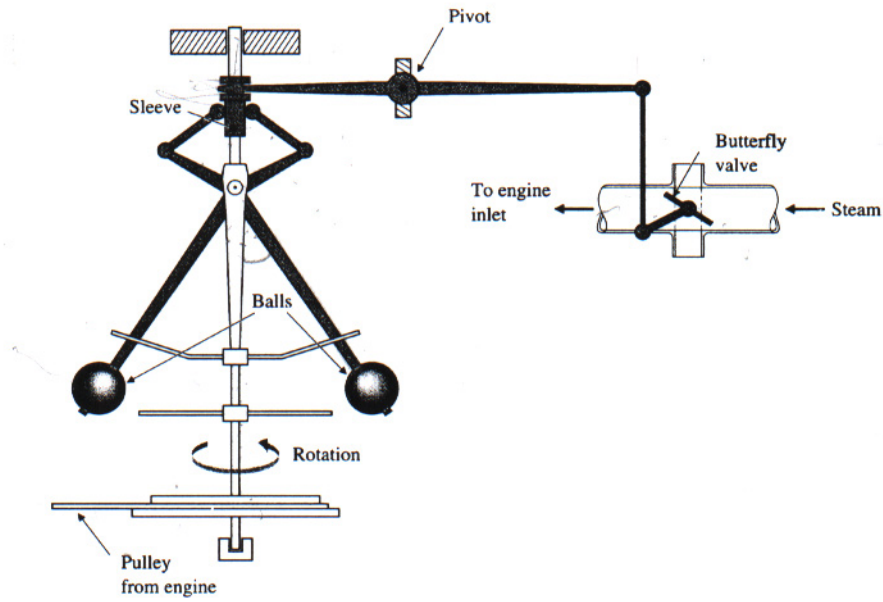
For $\Omega = 15.2$ rad/sec.



θ vs. t for $\Omega = 15.2$ rad/sec.



Watt's Governor

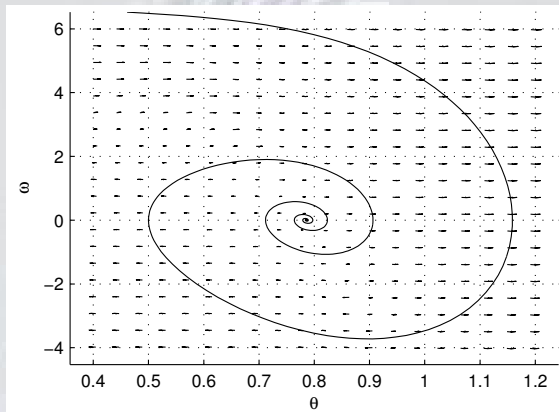


Now we have a governor that will maintain the desired angle but oscillates perpetually. How can we improve this performance?

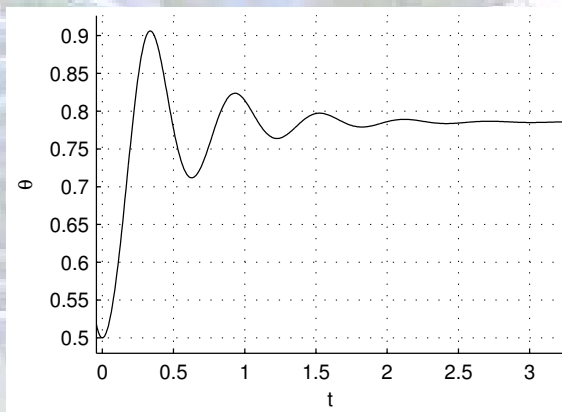
$$\ddot{\theta} = \Omega^2 \sin \theta \cos \theta - \frac{g}{R} m g \sin \theta - k \frac{\dot{\theta}}{m} \quad (6)$$

The damping term is proportional to the angular velocity (in the vertical plane) and is divided by the mass.



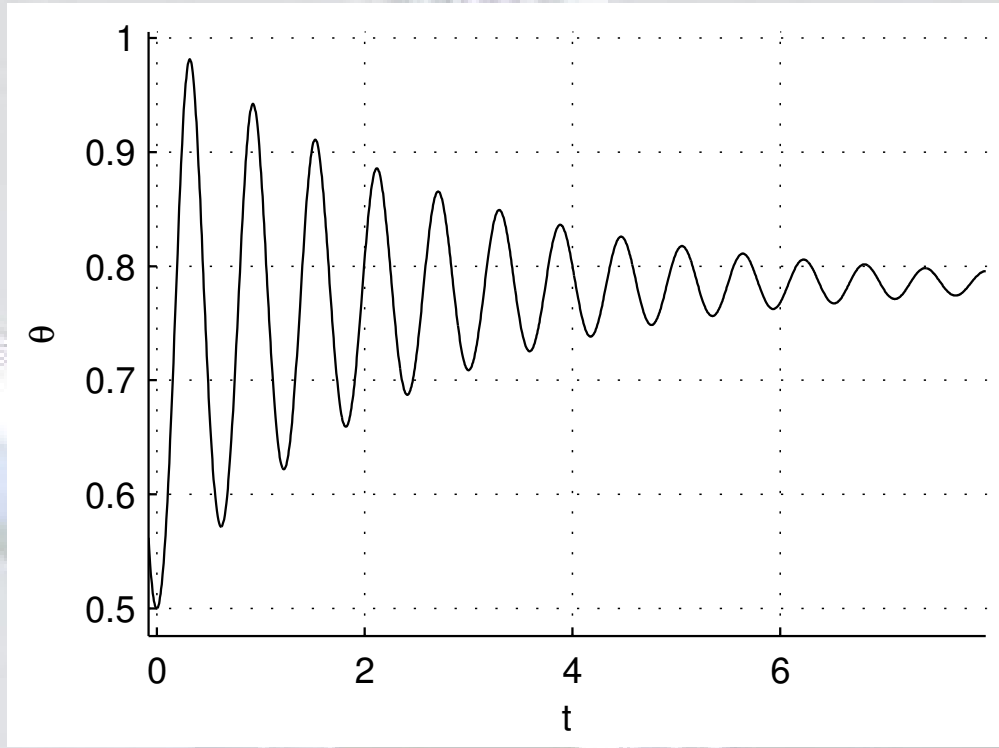


$\Omega = 15.2$ rad/sec with damping term.



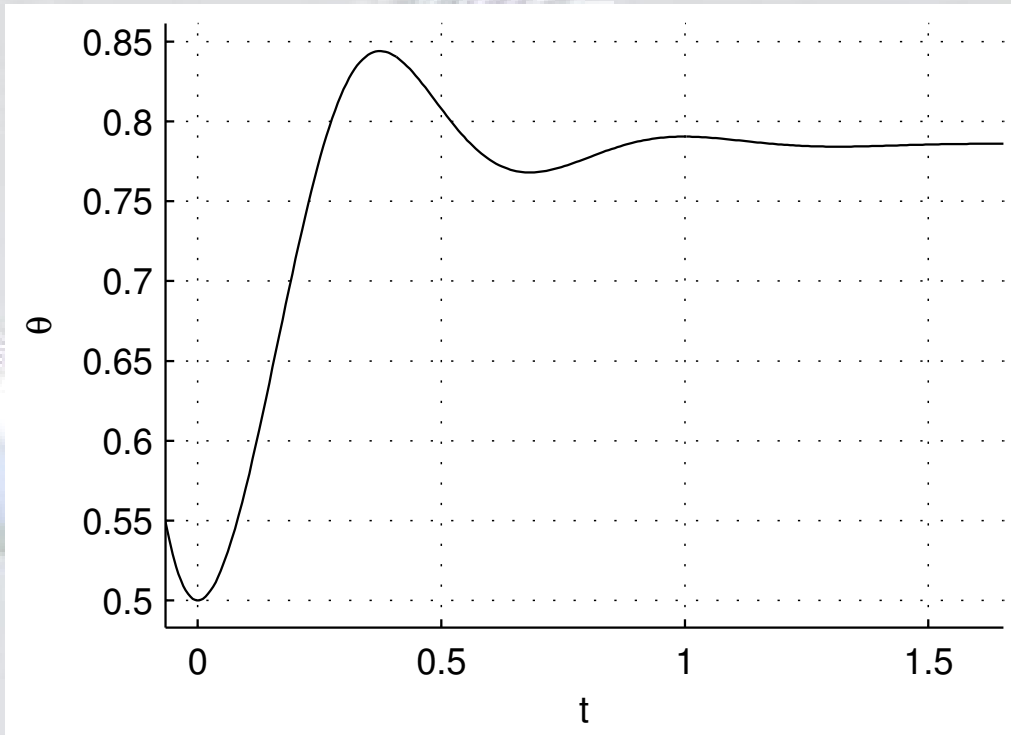
θ vs. t for $\Omega = 15.2$ and damping term.





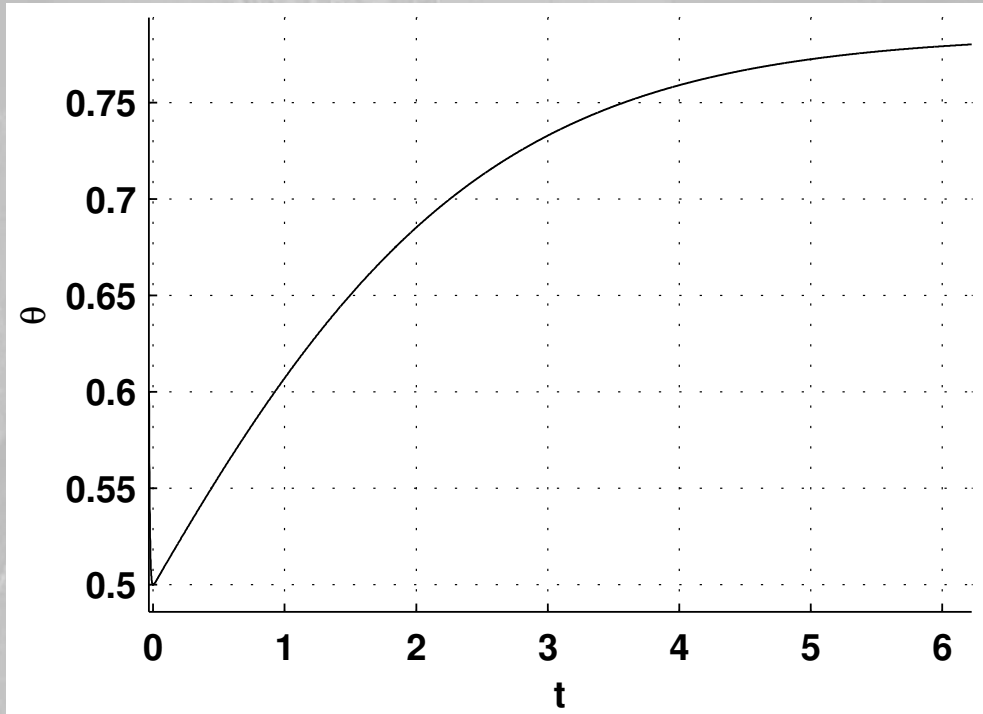
θ vs. t for $\Omega = 15.2$, damping term, and $m = 50g$.





θ vs. t for $\Omega = 15.2$, damping term, and $m = 5g$.





θ vs. t for $\Omega = 15.2$, damping term, and $m = .25g$.

As you can see this also would not be a governor of optimum design. When designing a governor one would have to experiment with the parameters and would undoubtedly be somewhere between $5g$ and $1/4g$.





Putting It All Together

We have a governor design that will maintain the desired

- Changing R only effects the where the critical Ω s occur but not the oscillatory behavior.
- Increasing the mass reduces the effects of damping, reducing mass increases the effects of damping.
- Changing the damping term has an inverse effect as changing the mass.

