Computers, Lies and the Fishing Season

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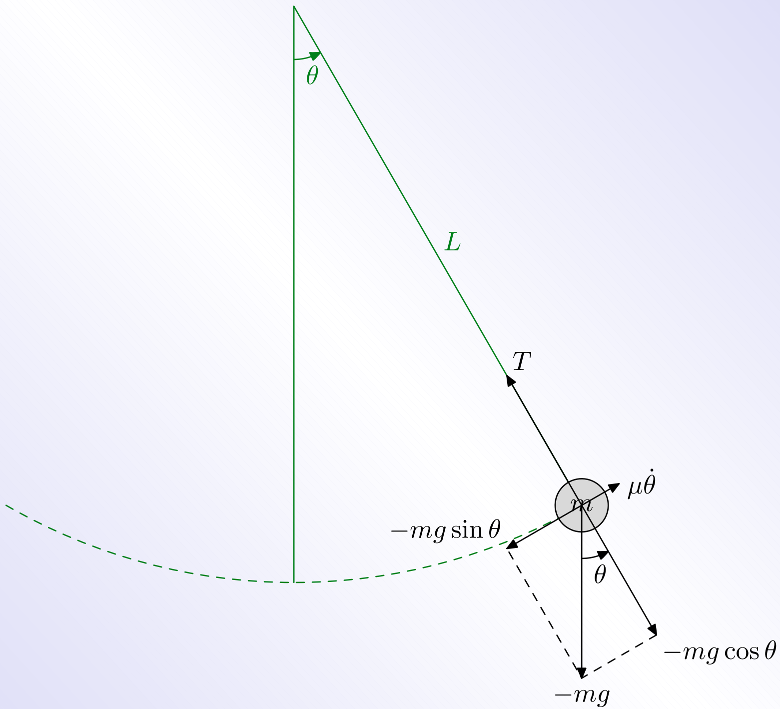
Introduction

Computers, lies and the fishing season takes a look at computer software programs. As mathematicians, we depend on computers for numerical answers. As we will see in this presentation, problems will occur in our computer computations. What you may think is the correct answer, is in fact not. We will consider two examples of the pendulum that illustrate this point. Then, we will consider two models of "harvested" populations and how the limitations of differential equation solver software may give "weird" or "incorrect" answers.

The Pendulum



Finding the equation for the pendulum

We know that

$$\theta = \frac{s}{r}.$$

However, in this case $r = L$, so $\theta = s/L$ and

$$s = L\theta.$$

Differentiating with respect to time (where L is constant),

$$\frac{ds}{dt} = L\frac{d\theta}{dt}$$
$$v = L\dot{\theta},$$

where v is the velocity.

Differentiating again with respect to time,

$$\frac{dv}{dt} = L\frac{d\ddot{\theta}}{dt}$$
$$a = L\ddot{\theta},$$

where a is the acceleration.

According to Newton's second law, the sum of the forces acting on an object equals its mass times acceleration. Thus,

$$ma = \sum F$$
$$ma = -mg\sin\theta - \mu L\dot{\theta} + f(t),$$

where $f(t)$ is the driving term. Substituting $a = L\ddot{\theta}$ we get

$$mL\ddot{\theta} = -mg\sin\theta - \mu L\dot{\theta} + f(t).$$

Dividing through by mL,

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\mu}{m}\dot{\theta} + g(t) \qquad (1)$$

Linearization

Now that we have the equation of a pendulum, let's look at the case where there is no driving term. Our equations now becomes

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\mu}{m}\dot{\theta}.$$

Notice that the term $-g\sin\theta/L$ makes this equation non-linear.

In fact, the $\sin\theta$ is the key term that is making this equation non-linear. If we assume that θ is small, we can use the approximation $\sin\theta \approx \theta$. Now our equation becomes

$$\ddot{\theta} = -\frac{g}{L}\theta - \frac{\mu}{m}\dot{\theta}. \tag{2}$$

Notice that our equation is now linear. So, for very small amplitudes of oscillations, the motion can be described by equation (2).

Example 1

Sampling Rates

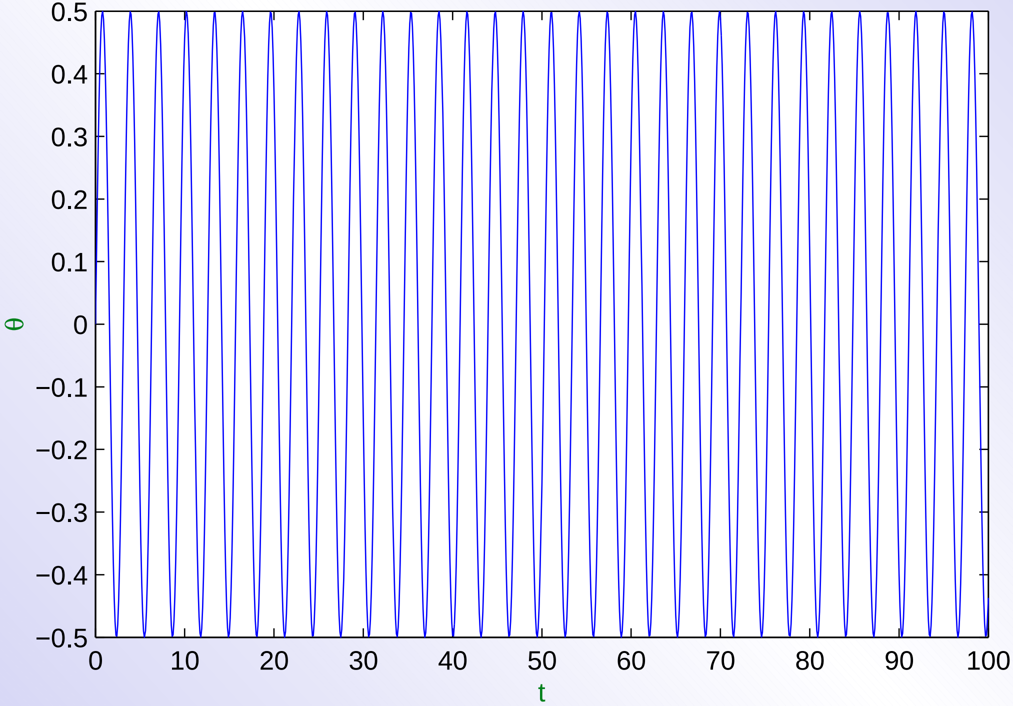
To begin with we are going to look at the linearized equation of a pendulum with no damping term involved. We are going to examine the simple harmonic oscillator equation,

$$\ddot{\theta} + 4\theta = 0, \text{ given that } \theta(0) = 0, \dot{\theta}(0) = 1.$$

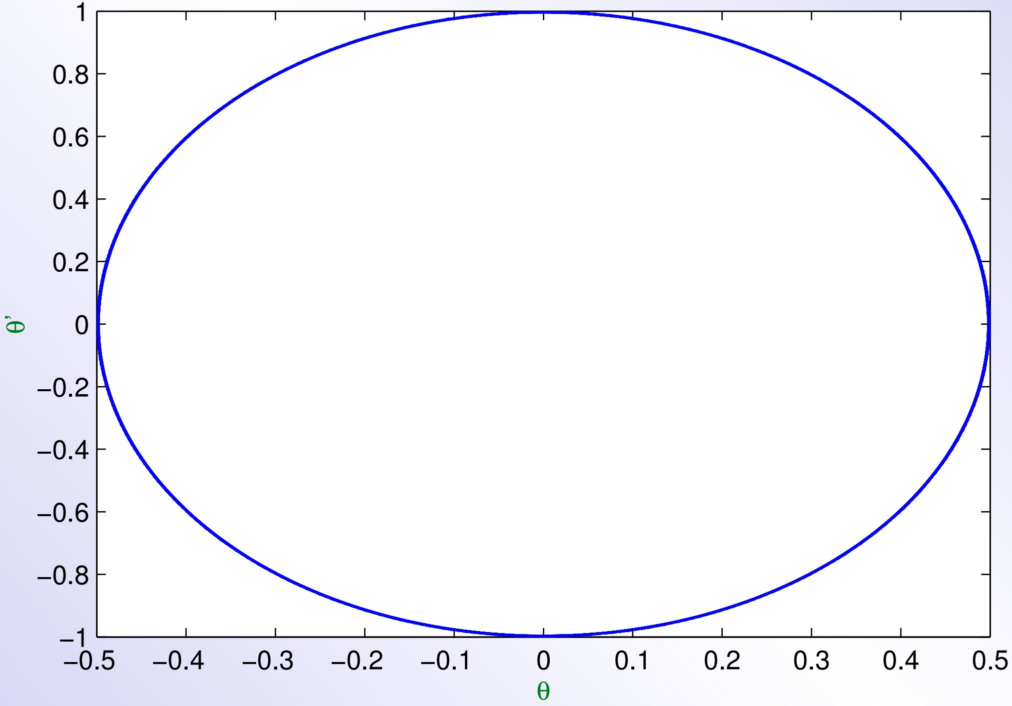
We are going to show 8 plots of the solution that satisfies the initial conditions. The interval $0 \leq t \leq 100$ was divided by 1000 equally spaced points and a fourth-order Runge-Kutta method was used to approximate the solution at these points. To graph the solutions, we are going to chose every point, every fifth point, every tenth point and every twentieth point. Each set of points will be plotted in the $t\theta$-plane and $\theta\dot{\theta}$-plane.

Plotting Solution Curves and orbits of the harmonic oscillator

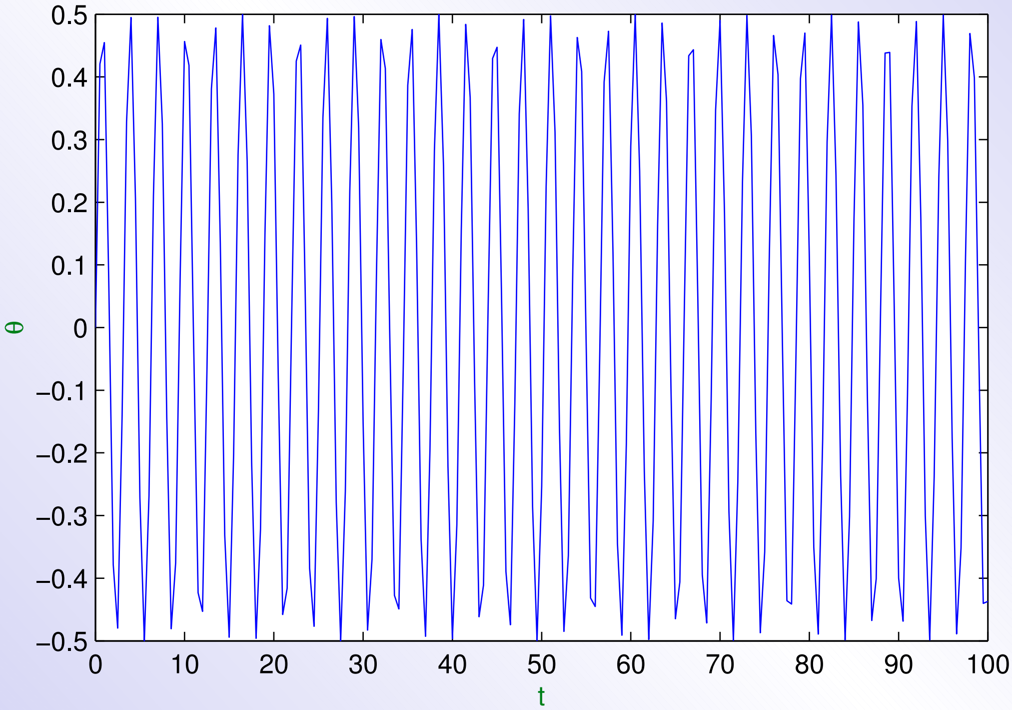
$t\theta$-plane using every point



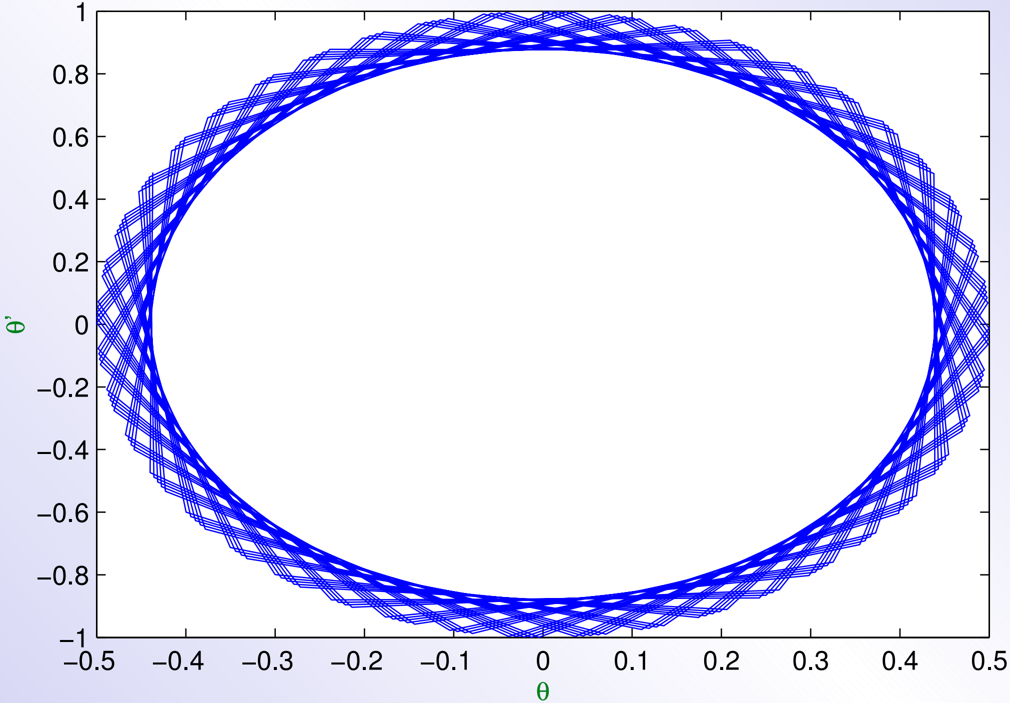
$\theta\dot{\theta}$-plane using every point



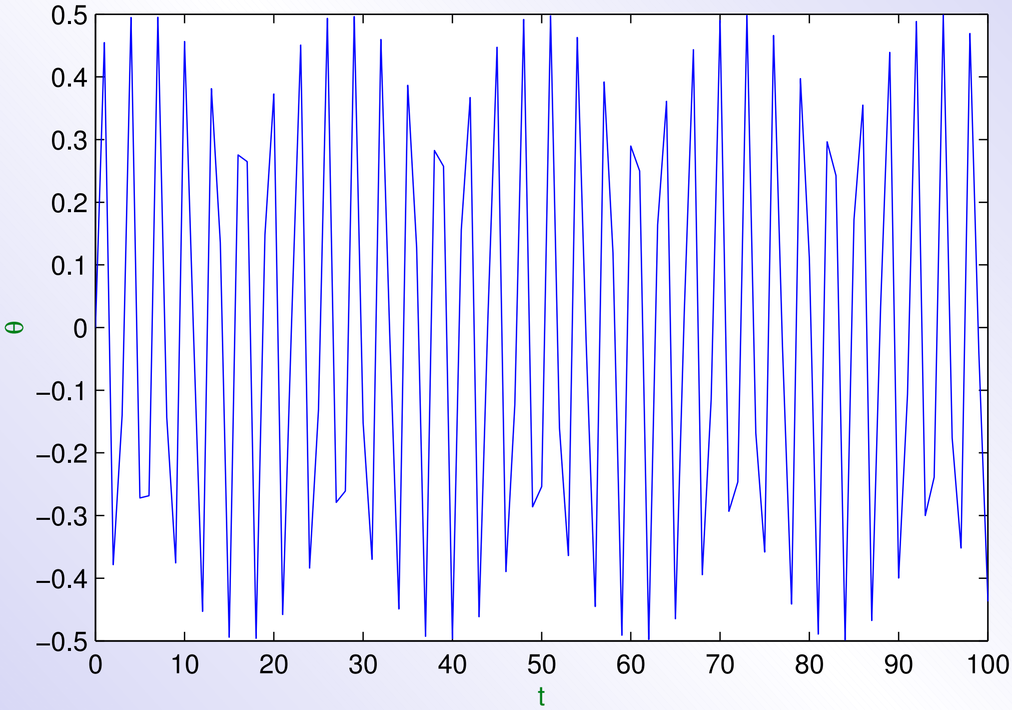
$t\theta$-plane using every fifth point



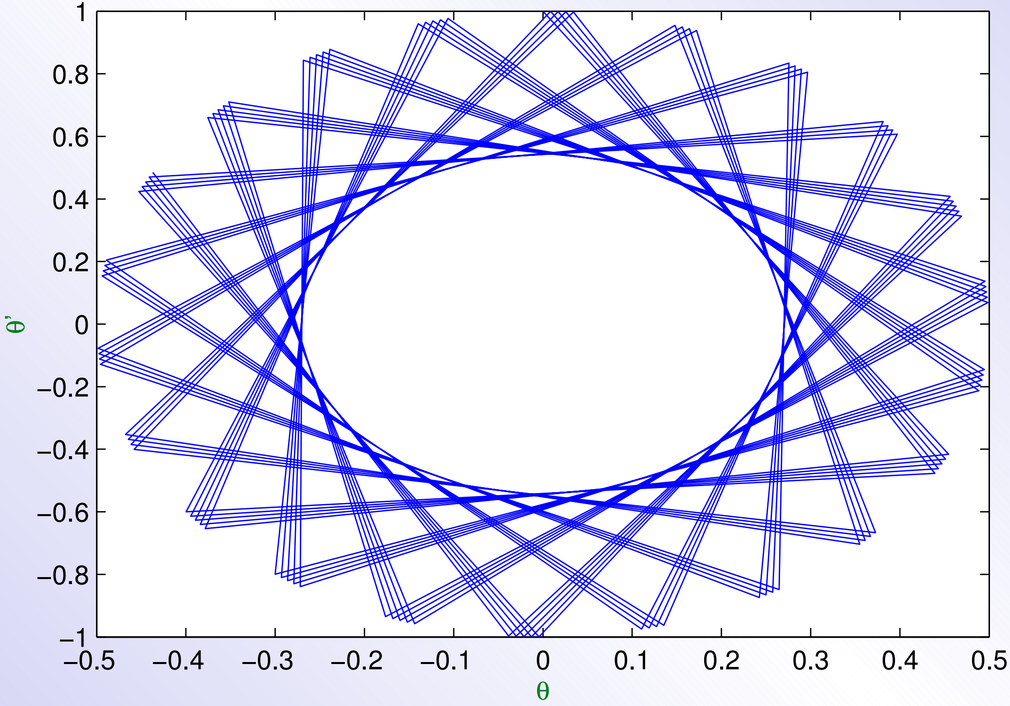
$\theta\dot{\theta}$-plane using every fifth point



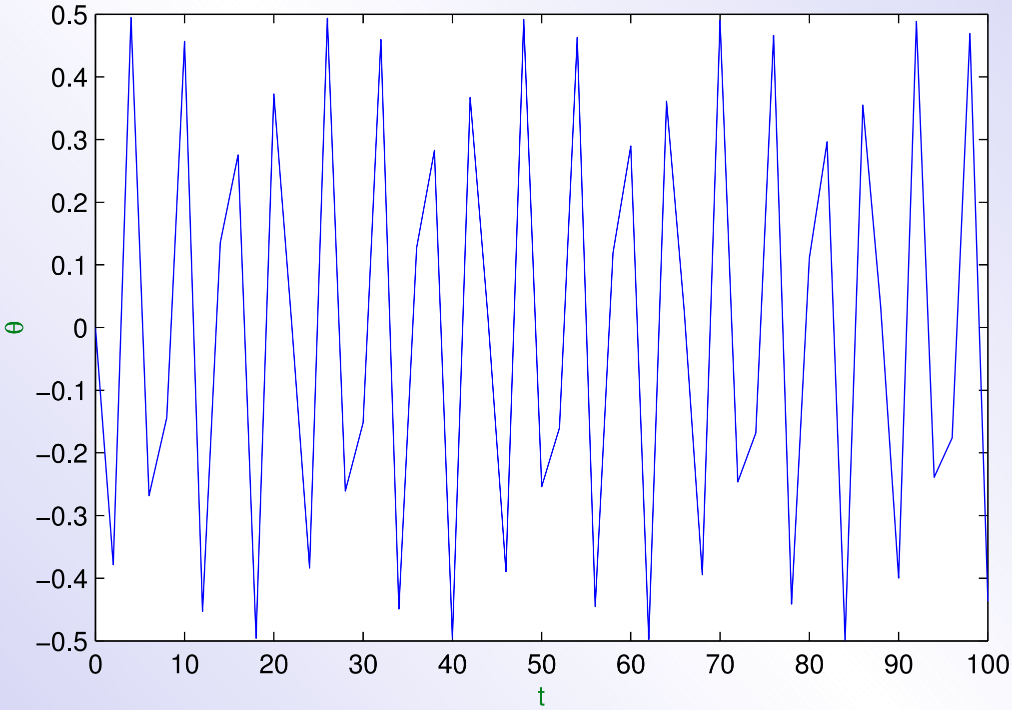
$t\theta$-plane using every tenth point



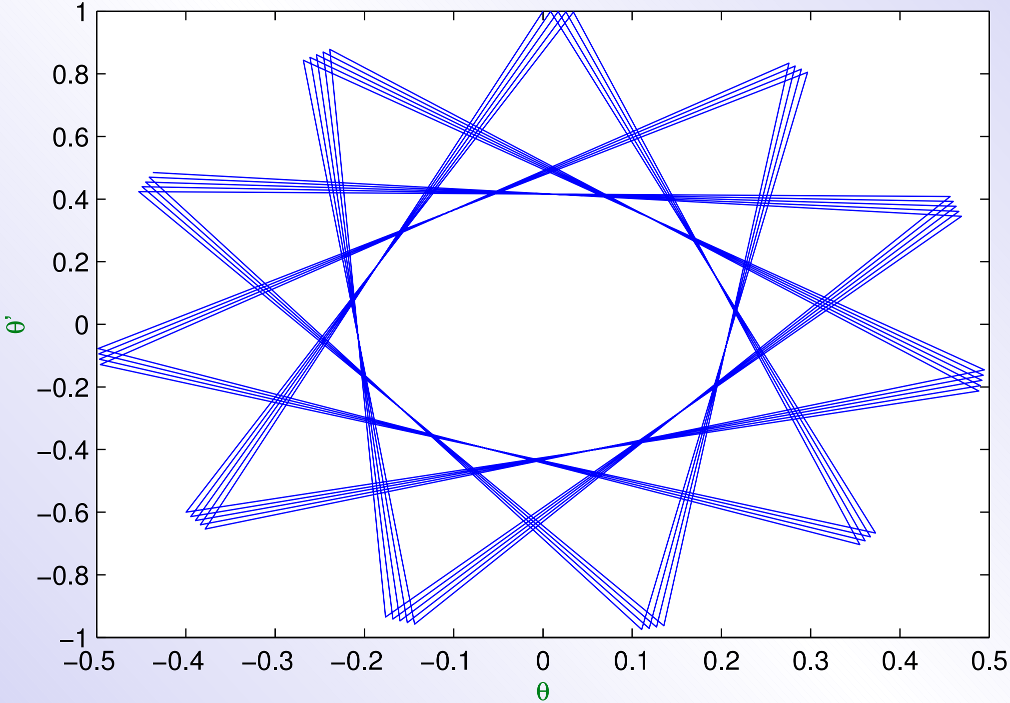
$\theta\dot{\theta}$-plane using every tenth point



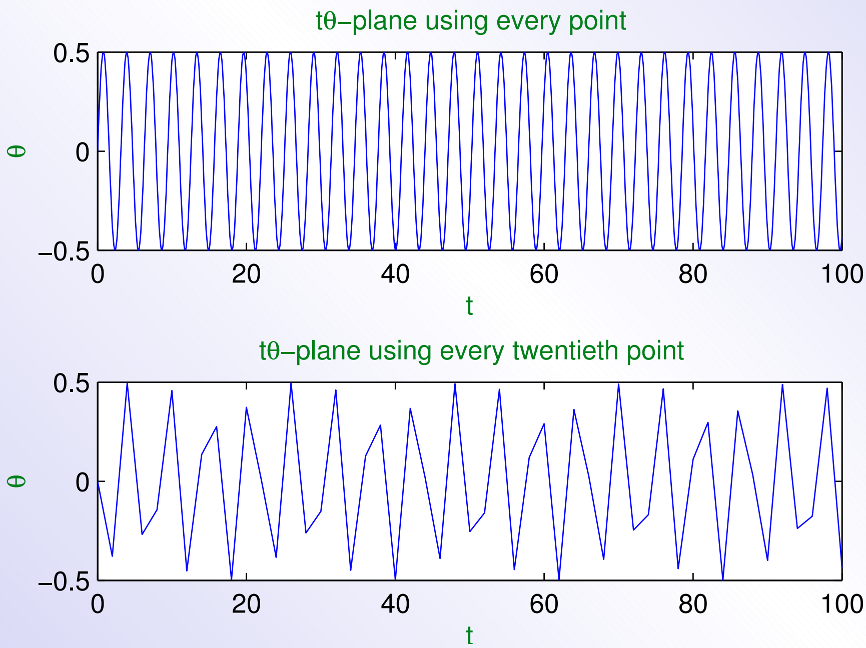
$t\theta$-plane using every twentieth point



$\theta\dot{\theta}$-plane using every twentieth point



As fewer and fewer points are plotted notice how the graphs are modulated. Also, notice that the period of the graphs plotting every point, every fifth point and every tenth point seem to have the same value of π, but the graph of plotting every twentieth point seems to have doubled to 2π.



What is going on here? Is the accuracy of the solver to blame? No. The accuracy of all the points taken are just as good as the other ones. All of the points were sampled from the same list of 1000 approximate values. What we observed from looking at the graphs always occurs when an oscillatory curve is approximated by a finite number of equally spaced points connected by a straight line segment. As the number of sampled points per unit time decrease, the graphs appear to worsen.

Aliasing

Aliasing occurs when there is not enough discrete points to reconstruct the shape of the original graph. Notice how the graph looks when plotting every point. But when you sample fewer than two sample points per period as in the graph plotting every twentieth point, the graphical representation appears to have a larger period.

Conclusion

What can we do to correct the misrepresentation in representing a continuous periodic function by a discrete point set? If the experimenter knew before hand, the period of oscillation, then the "sampling rate" could be set high enough to decrease the problem.

Example 2

The Driven Pendulum

We have just examined the equation of a pendulum with no damping term. Now we are going to look at the equation of a pendulum with a damping term involved. Recall that the equation for a pendulum with a damping term is

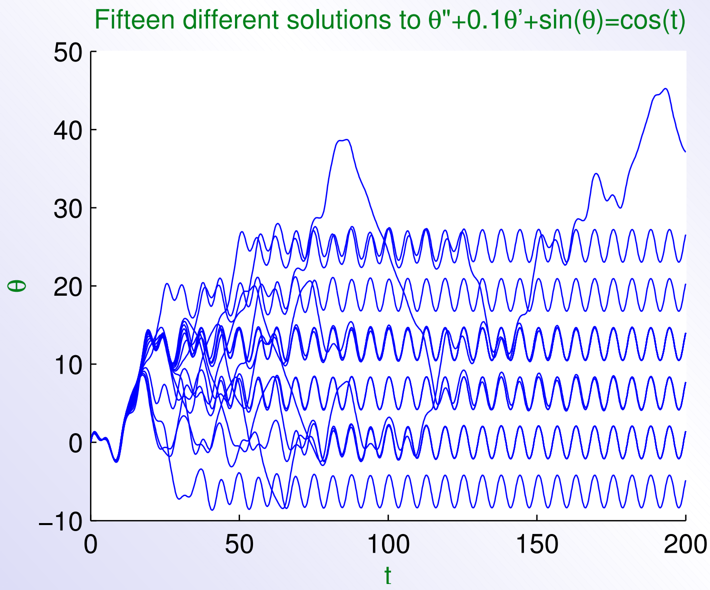
$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\mu}{m}\dot{\theta} + g(t).$$

The equation that we are going to examine is

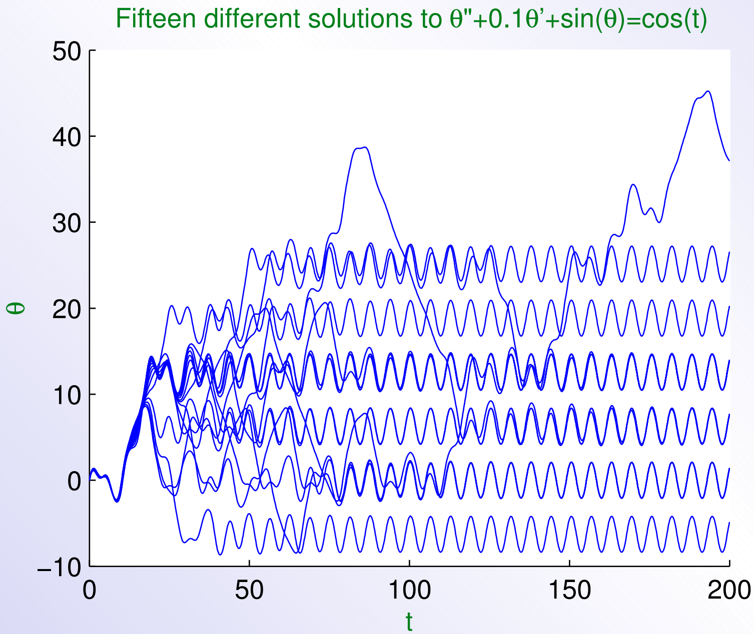
$$\ddot{\theta} = -\sin\theta - 0.1\dot{\theta} + \cos t.$$

What Does The Motion Look Like?

This picture shows the motion of our pendulum resulting from 15 different sets of initial angular velocities. Each solution starts with the position $\theta(0) = 0$, while the initial angular velocities are evenly spaced between 1.85 and 2.1. The graph is plotted on the intervals $0 < t < 200$ and $-10 < \theta < 50$.



Notice how the 15 solutions are different from one another even though they have the same starting position. The initial angular velocities affect each solution differently. The difference in the initial angular velocities is only 0.25 and the solutions behave in this way. Imagine how the solutions would look like if the difference between the initial angular velocities was greater than 0.25.

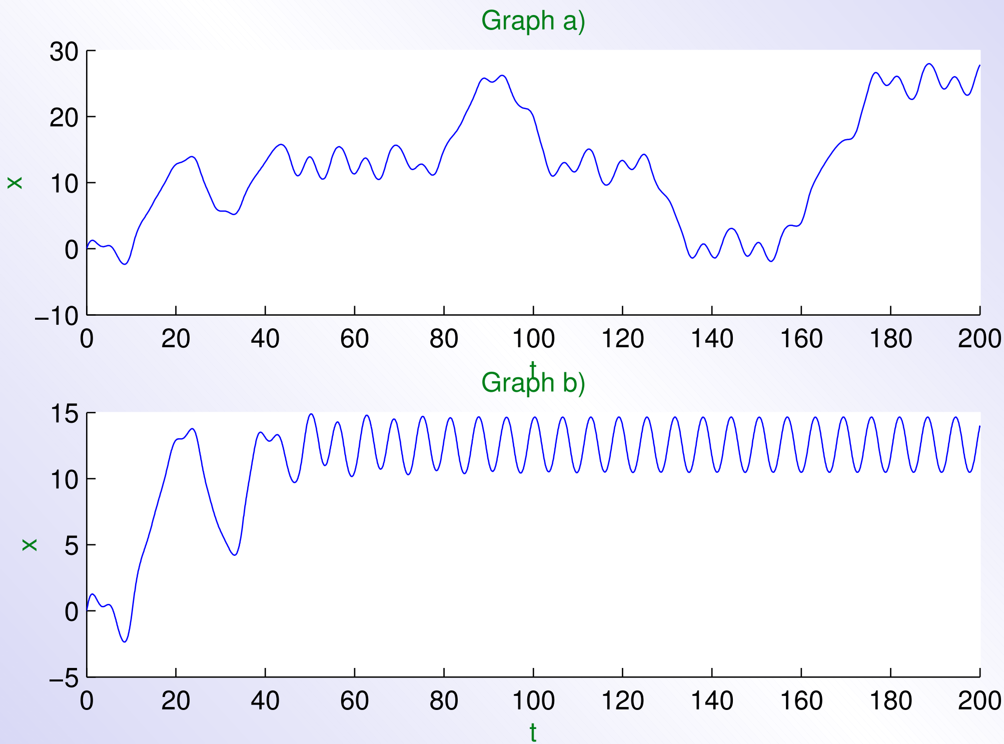


We saw what happens to our damped pendulum when you change the initial angular velocities slightly. Now, let's look at the case we where we reduce the relative and absolute local error tolerance. We are going to continue to use the equation $\ddot{\theta} = -\sin\theta - 0.1\dot{\theta} + \cos t$, with the initial position of $\theta(0) = 0$ and the initial angular velocity of $\dot{\theta}(0) = 2$.

Graph a) is the graph of the pendulum with an absolute local error tolerance of 4×10^{-4} and a relative local error tolerance of 4×10^{-4}.

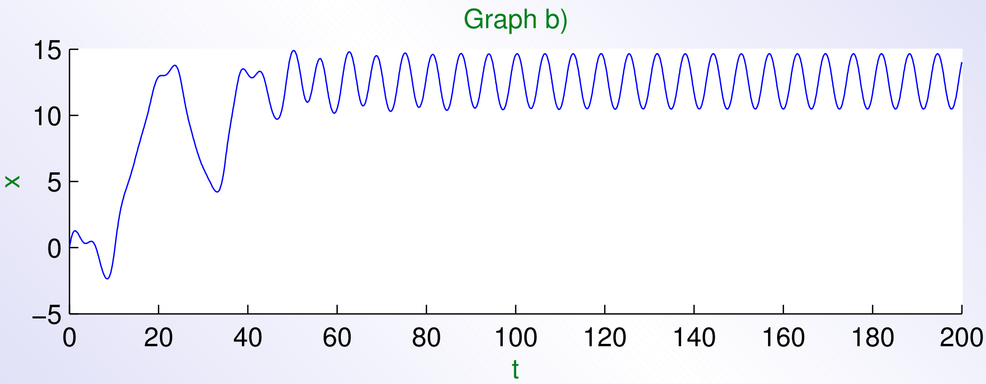
Graph b) is the graph of the pendulum with an absolute local error tolerance of 4×10^{-6} and a relative local error tolerance of 4×10^{-6}.

Notice how graph a) is settling down at about 25 while graph b) is settling down at about 12.



Which Graph Is The Correct Solution?

Just by reducing the relative and absolute local error tolerance, you see that we get two completely different answers for the same problem. Which one is the correct answer? The answer is the graph with the smaller relative and absolute local tolerances, which is graph b).



When graphing these, you should experiment with different values of the relative and local error tolerances until you get two graphs that look similar and then you know that the graph you have is the correct solution.

Conclusion

As we saw, when graphing a damped harmonic oscillator, you can have different solutions just by increasing or decreasing the relative and absolute local tolerances. You must experiment with different values for relative and local error tolerances until you are satisfied that you have a correct representation of the graph.

Example 3

The Logistic Model With A Fishing Season

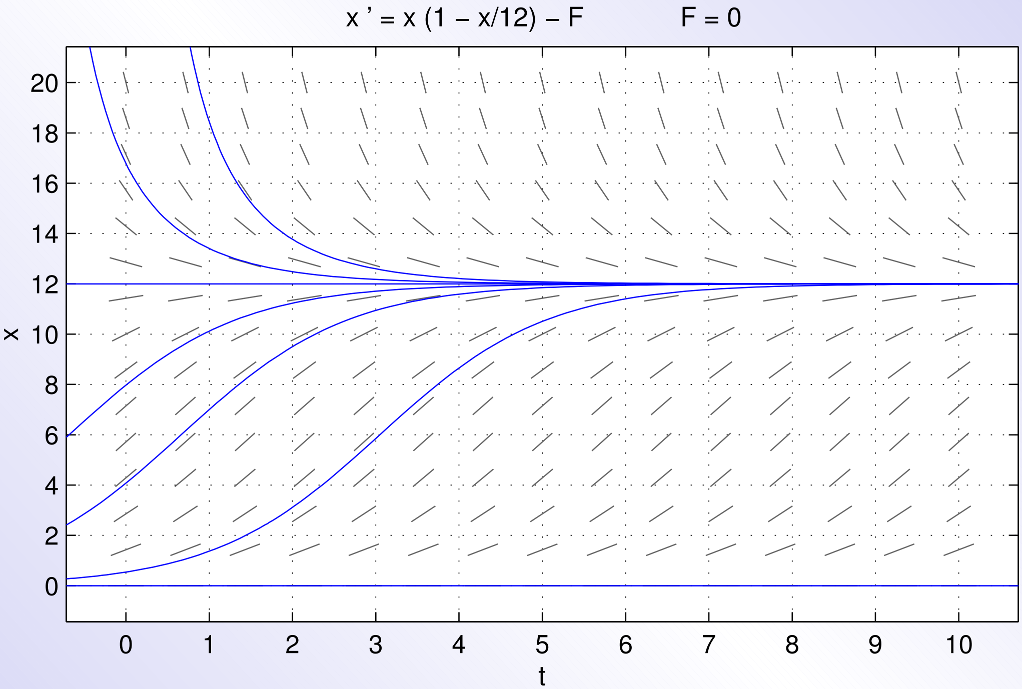
We are now going to look at model for a bounded population with a harvesting rate. The best known equation for this model is

$$x' = rx\left(1 - \frac{x}{K}\right) - F,$$

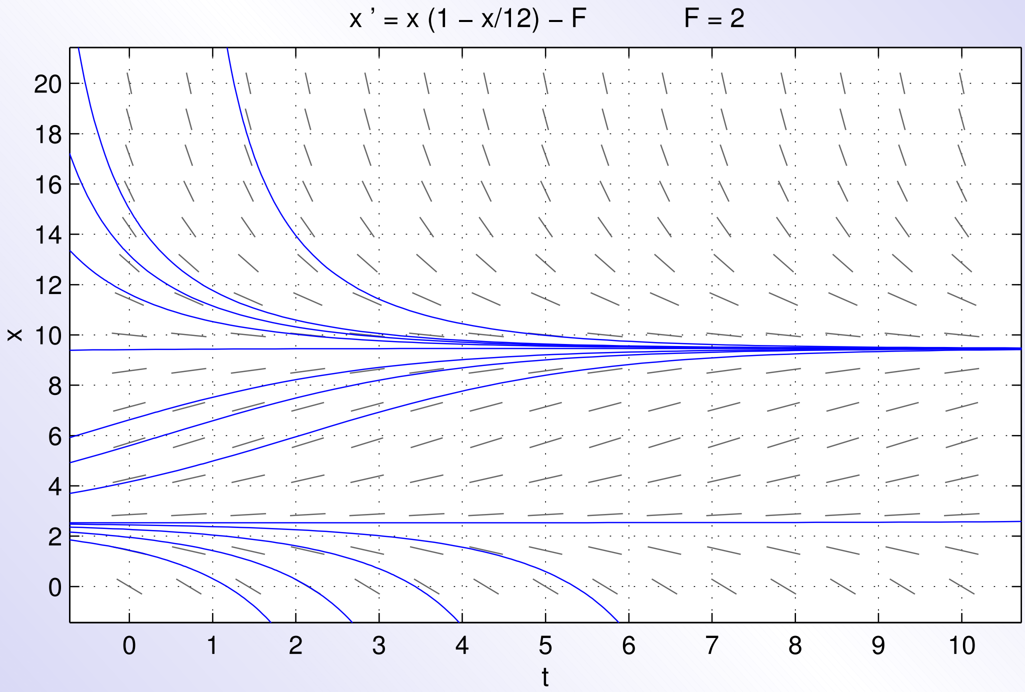
where $x(t)$ is the population at time t, r, and K are positive constants, and F is the harvesting rate. $x(t)$ is measured in tons, and t is measured in years. The equation that we are going to examine is

$$x' = x(1 - x/12) - F.$$

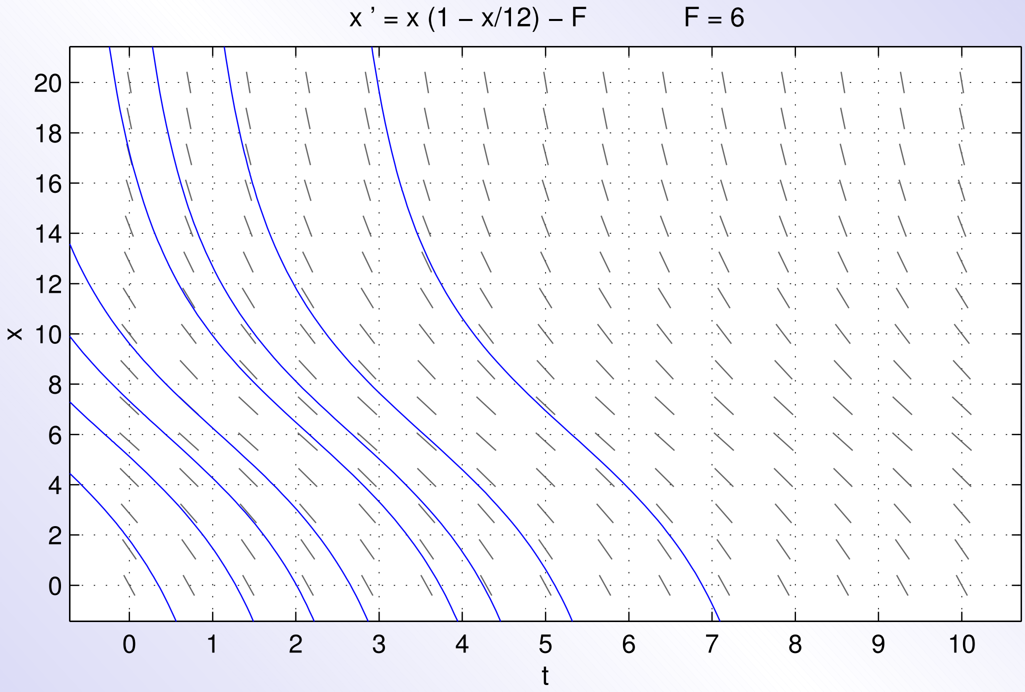
If there is no fishing $(F = 0)$, then the fish population tends to the equilibrium level K as shown.



Continuous fishing at a constant but low, rate $(F = 2)$, leads to a stable equilibrium below K as shown. Notice that if the initial population level is too low, extinction for the fish is inevitable.

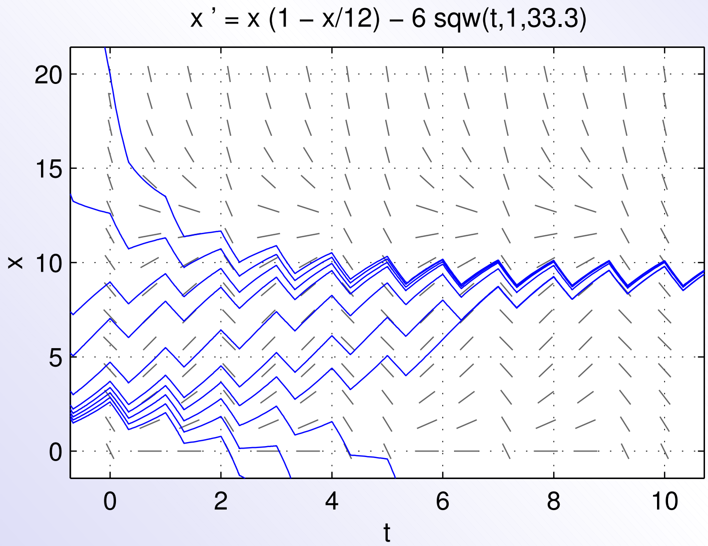


Continuous fishing at a high constant rate $(F = 6)$, leads to extinction eventually, regardless of the initial condition.

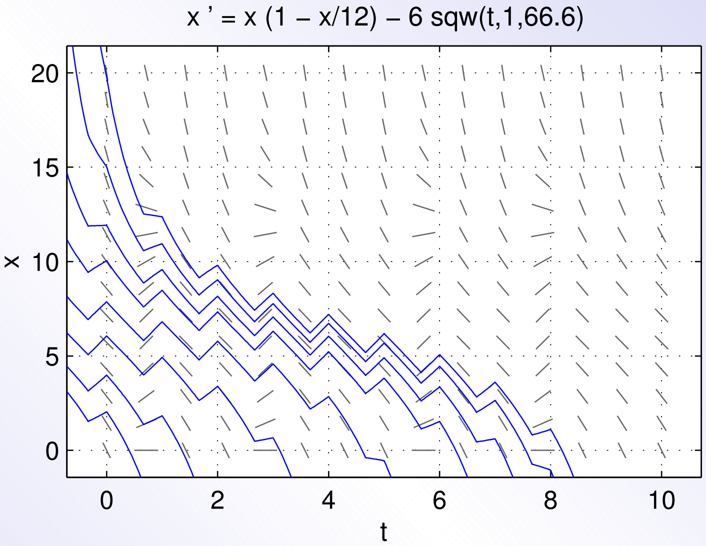


A high fishing rate will not lead to extinction if the fishing season is restricted to a few months each year. Using the high fishing rate equation $x' = x(1 - x/12) - 6$, look what happens to the fish population maintained over a four-month fishing season as well as an eight-month fishing season.

Four-month fishing season



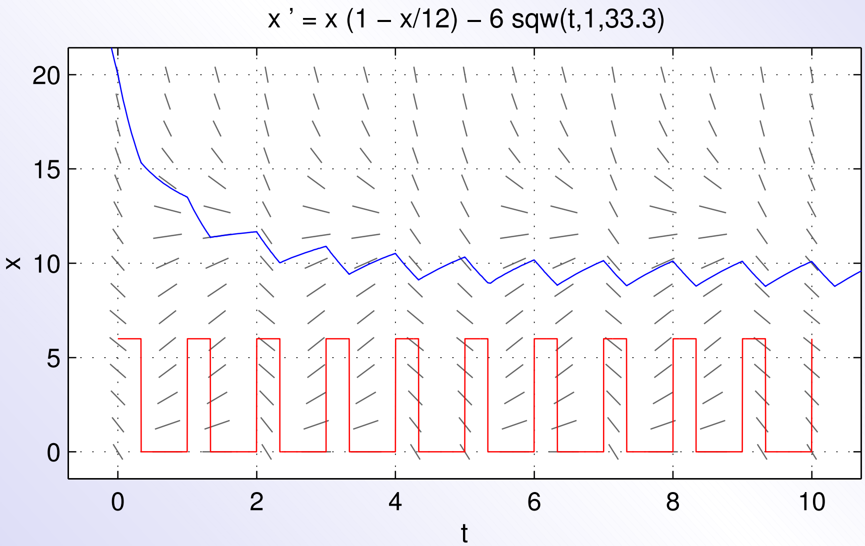
Eight-month fishing season



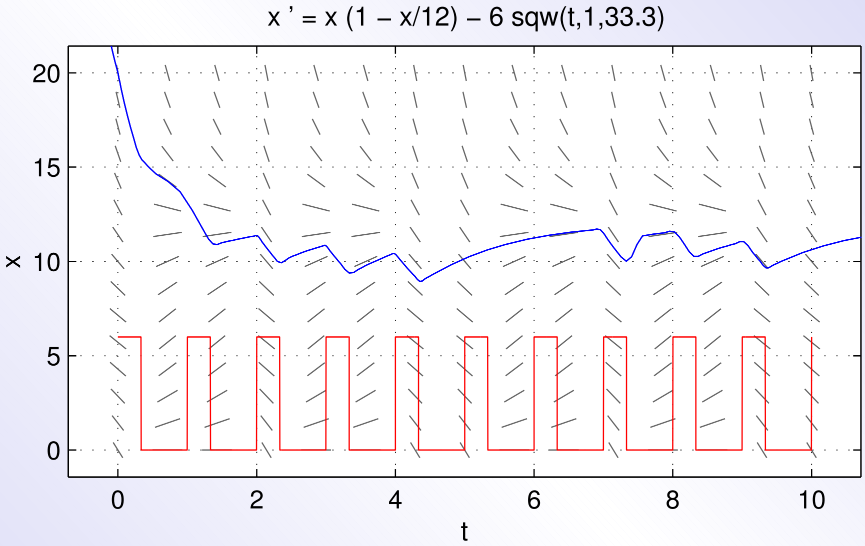
As you see when having a high rate of fishing and a four-month fishing season, the fish population will continue to remain if there are more than 3.2 tons of fish. For the case of an eight-month fishing season, you see that the fish population is headed for extinction no matter what.

Computer Lies

How good are the computed zigzag solution curves in out graphs that we just saw? The zigzag solutions are are pretty accurate if the maximal step size of the solver is kept small enough so that the solver recognizes the points in time when the harvesting function F switches on or off.

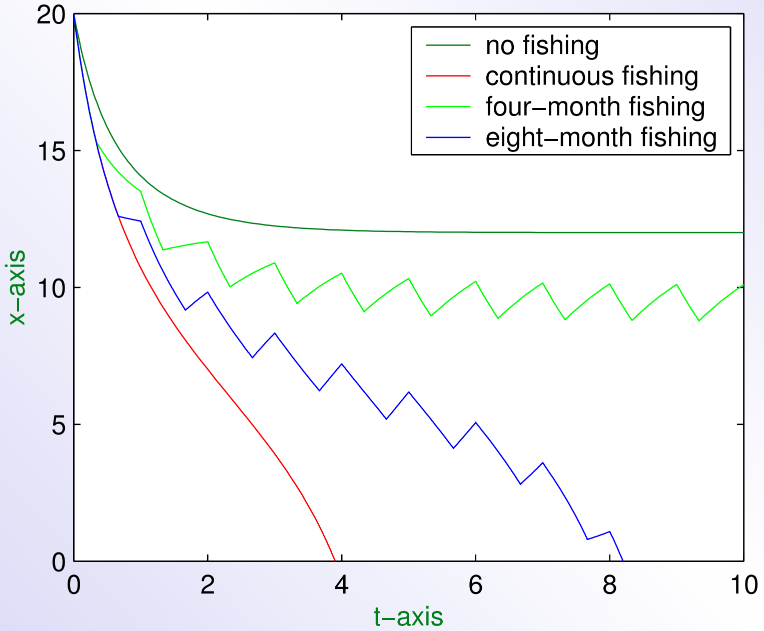


If the maximal step size of the solver is made large enough, then the solver will not be able to recognize the points in time when the harvesting function F switches on or off.

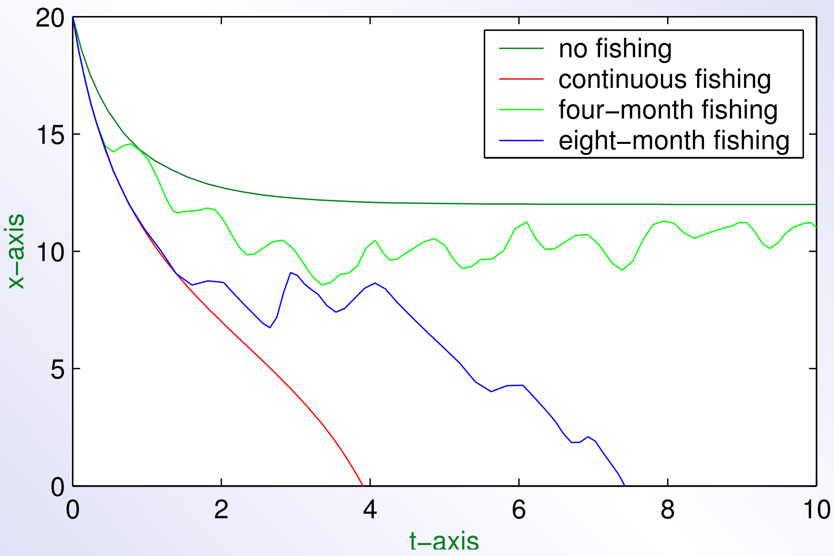


Alternatively, a large maximal step size is allowed if the local error bounds of the solver are set to very small levels.

This graph, reading from top to bottom, shows with accuracy the consequences of no fishing, a four month fishing season, an eight month fishing season and continuous fishing on an initial population of $x(0) = 20$ tons of fish. The relative local error tolerance is 1×10^{-7} and the absolute local error tolerance is 1×10^{-10}.

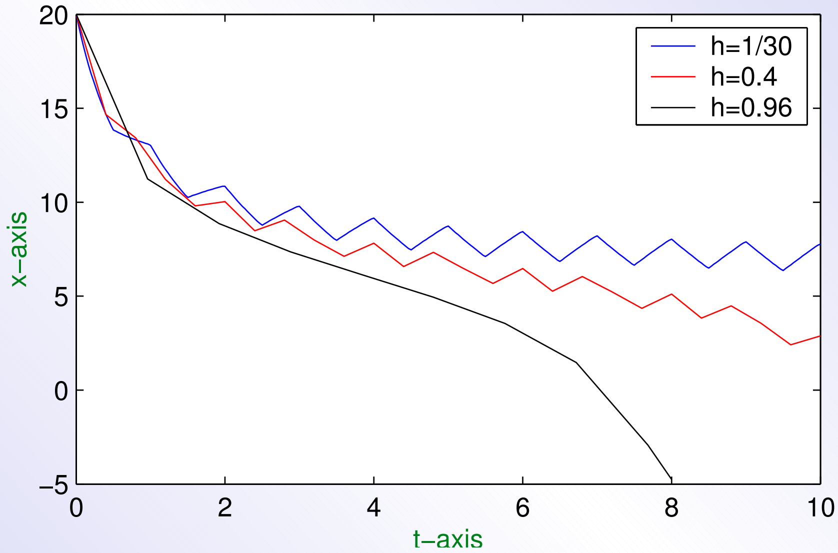


Now, when we increase the local error bounds of the solver, notice what happens to our solutions. The relative local error tolerance is increased to 1×10^{-2} and the absolute local error tolerance is increased to 1×10^{-2}.



You can see that the this graph is telling “lies” about the fishing season.

What happens when changing the step size of the solver set, fourth-order Runge-Kutta? The computer tells more “lies.” When the step size is changed, you get different solutions as shown.



Conclusion

When working with computers and graphs, keeping the step size of your solver small enough will help the solver recognize the points in time that are important and that will lead to accurate solutions. Also, if you have a large step size, then setting the local error bounds low enough will lead to accurate solutions as well. If neither of these conditions are met, then your computer solver may tell "lies" about your solution.

Example 4

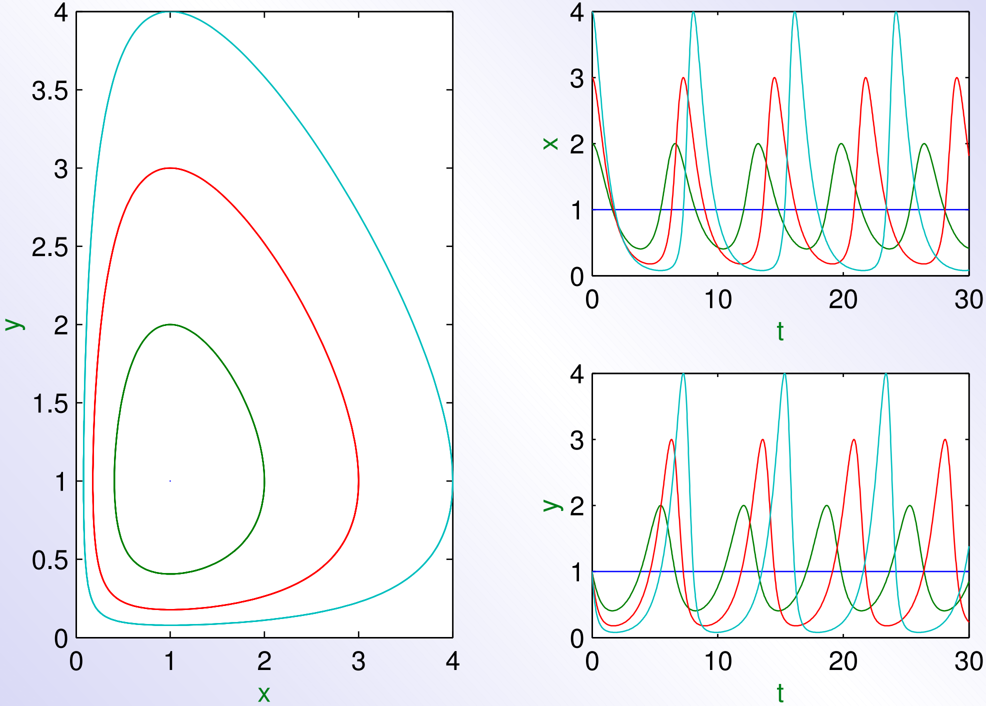
Predator-Prey Dynamics With Fishing

Finally, we are going to look at a predator-prey system of equations. The best known system of equations for the predator species $x(t)$ and the prey species $y(t)$ are:

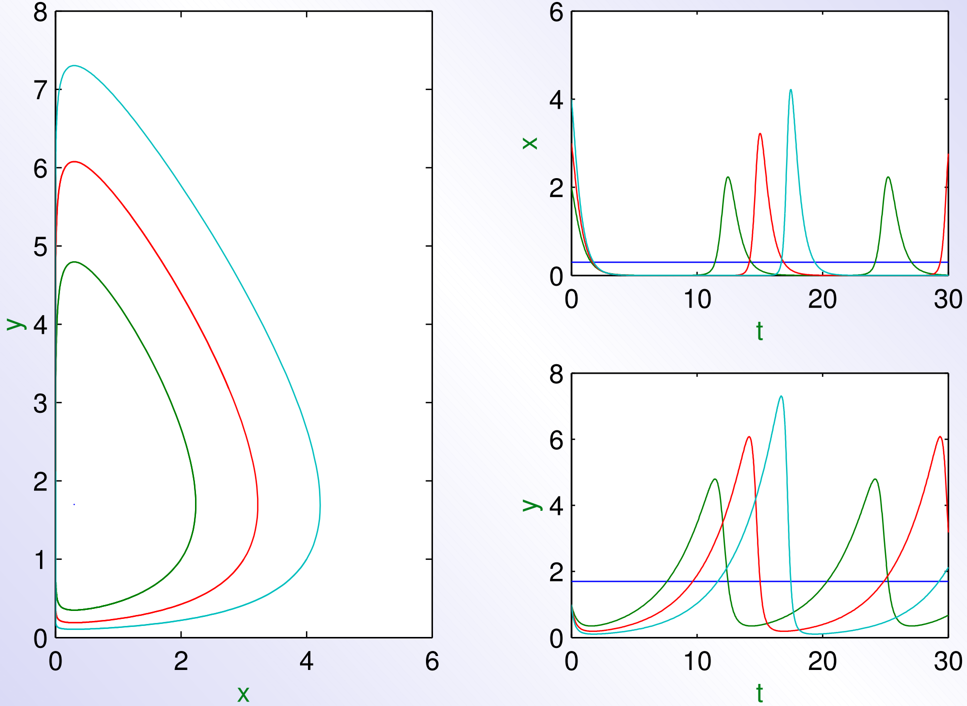
$$x' = ax(-1+by) - H_1 x$$
$$y' = cy(1-dx) - H_2 y$$

where a, b, c, d are positive constants and H_1 and H_2 are nonnegative "harvesting" coefficients. The harvesting terms $H_1 x$ and $H_2 y$ model constant effort harvesting rather than the constant rate harvesting that we examined previously.

We are going to plot the solutions of $x' = x(-1+y)$, and $y' = y(1-x)$, on the xy-plane, tx-plane, and the ty-plane. These equations examine no harvesting.

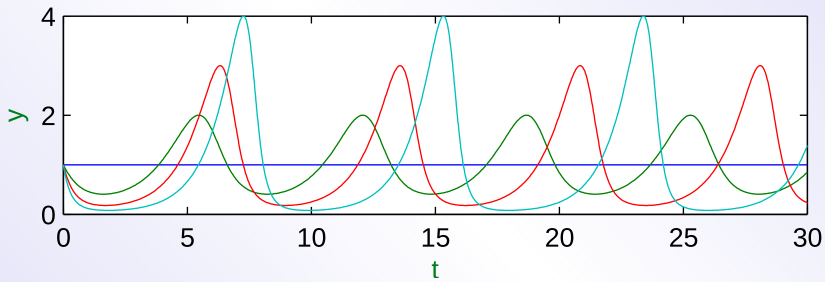


Now, we are going to plot the solutions of $x' = x(-1 + y) - 0.7x$, and $y' = y(1 - x) - 0.7y$, on the xy-plane, tx-plane, and the ty-plane. These equations examine continuous light fishing.



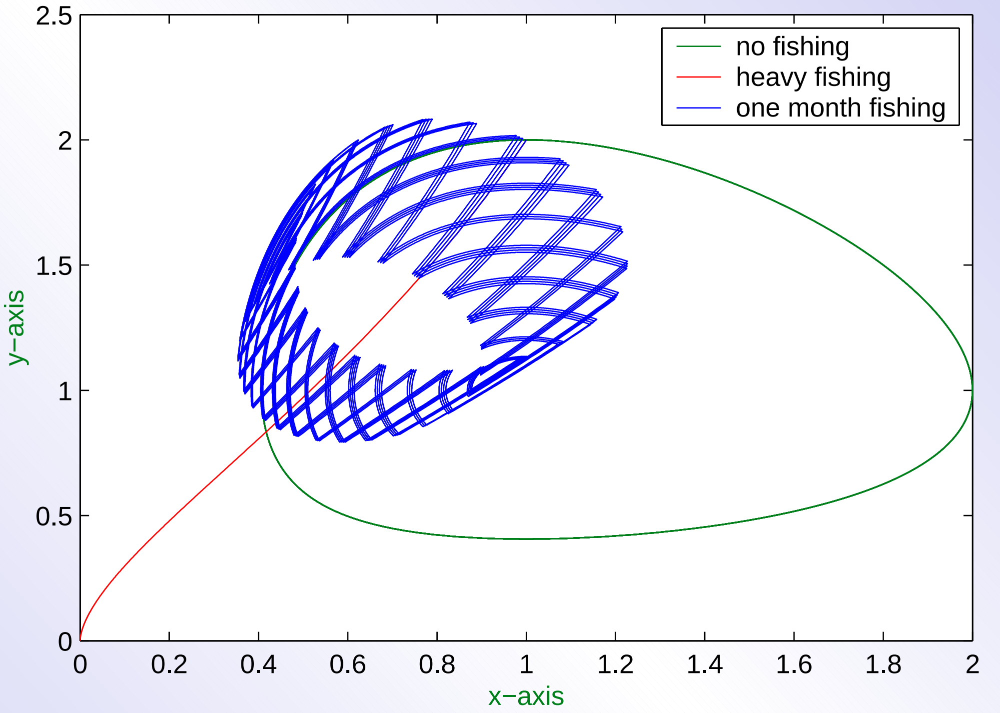
Nonlinear Phenomenon

A distinctly nonlinear phenomenon is occurring with the graph containing no harvesting. When the periods of the solutions increase with the amplitude of the cycle, a nonlinear phenomenon has occurred.



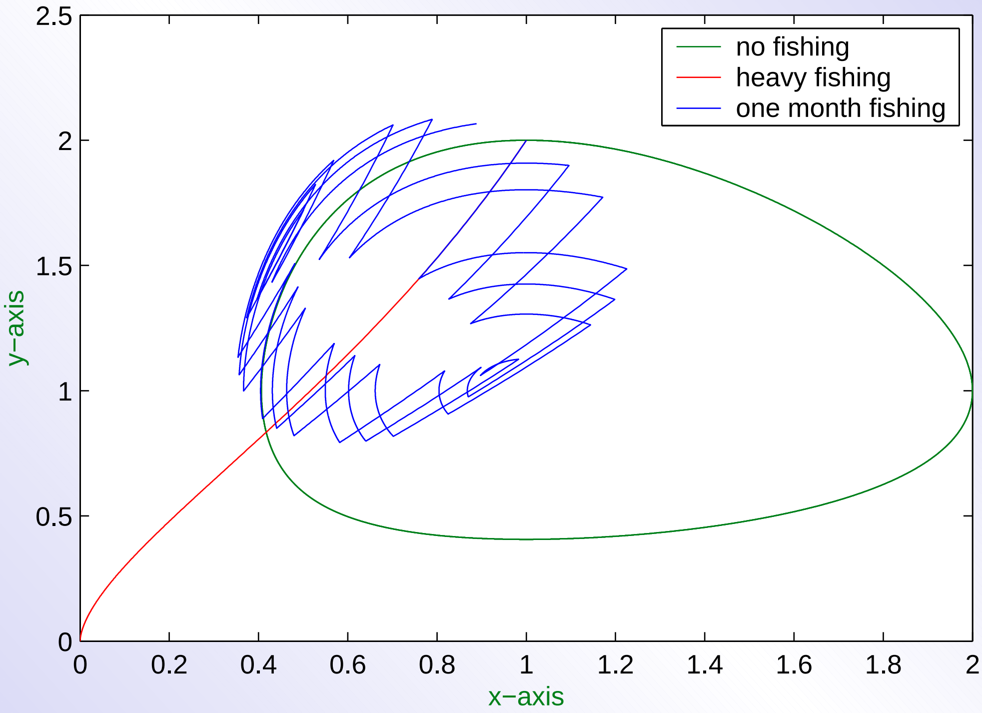
Seasonal Harvesting

Now, we are going to show what happens to a particular population cycle if both species are harvested with the high harvesting coefficients, $H_1 = H_2 = 4$, but with a short, one month harvest season per year. The graph shows an oval orbit of the no fishing case and an extinction orbit of the constant, heavy fishing case, and an orbit of fishing one month per year. The maximal step size is 0.01 and the relative local error tolerance is 1×10^{-7} and the absolute local error tolerance is 1×10^{-10}. The time span is 100 years and the number of points plotted is $10,000$.

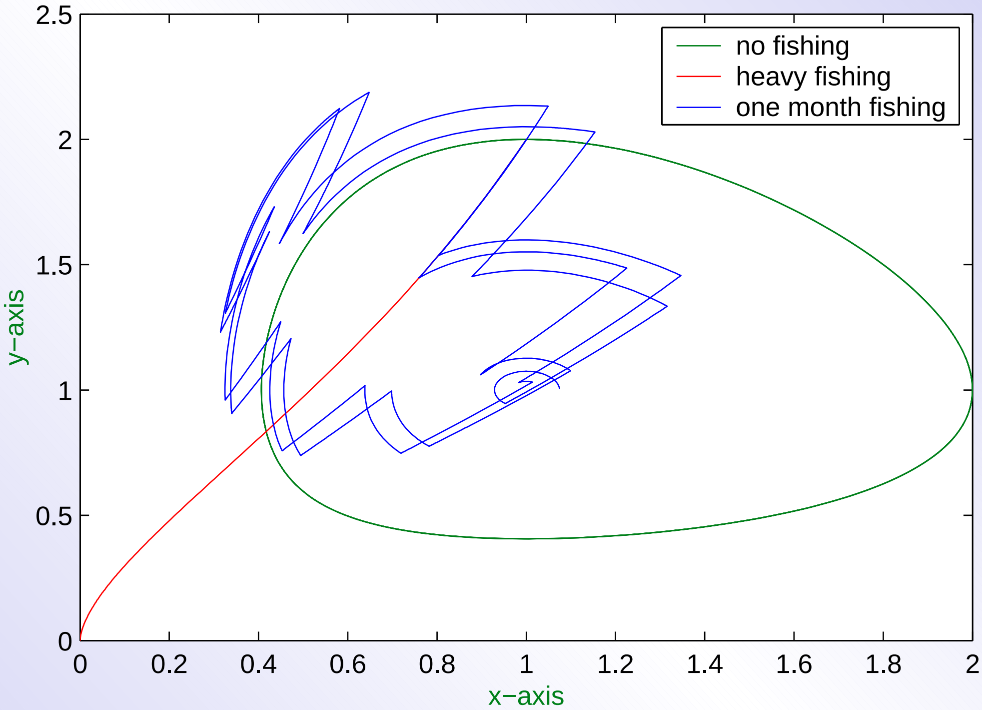


Each arc of the "bracelet" is an arc of one of the population ovals in the no-harvest case or an arc of an extinction curve in the continuous high-rate harvest case.

If we reduce the time span to 20 years and the number of plotted points to 2000, we get:

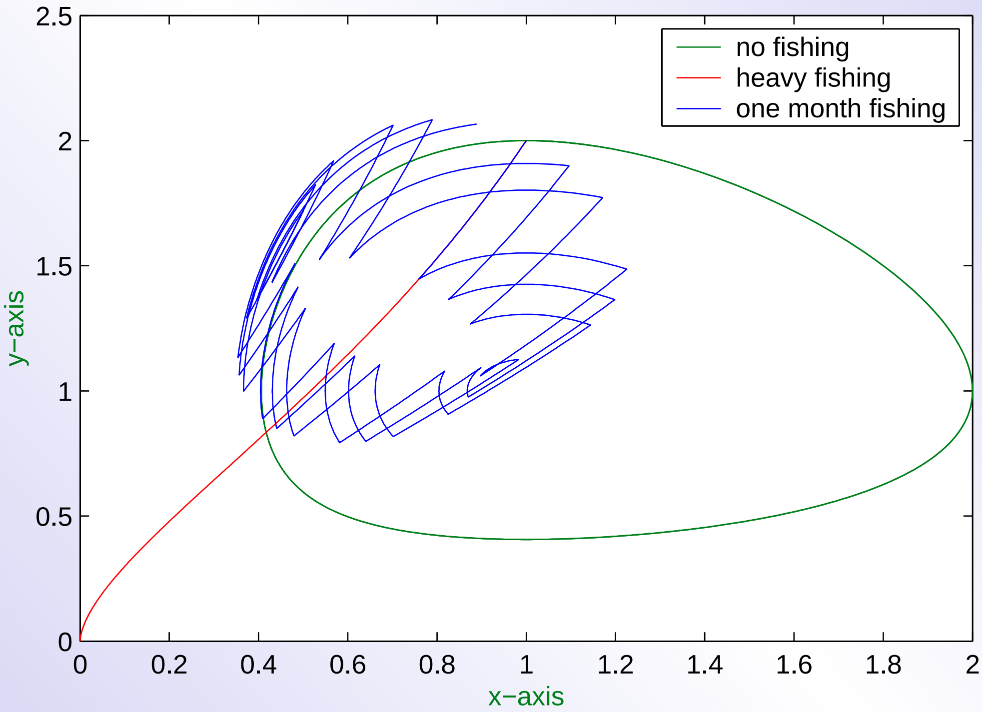


If we increase the step size to 0.20, we get:



The computer is now telling lies about the population. The maximal step size is too large so the solver can't "see" the sudden breaks in the harvesting coefficient when they occur so the solver produces this graph.

If we now leave the step size at 0.20, and decrease the relative absolute local tolerance to 1×10^{-9}, and decrease the absolute local tolerance to 1×10^{-12}, we get:



Conclusion

As we have seen in the case of the harmonic oscillator, the driven pendulum, the logistic model with a fishing season and the predator-prey dynamics with fishing, what you may think is a correct answer is in fact not. Every differential solver has is "kinks," so the user must experiment with their equations. You should not rely on the first graph that you see because the graph could be a lie.