Applications of Differential Equations

Fun With The Brachistochrone!

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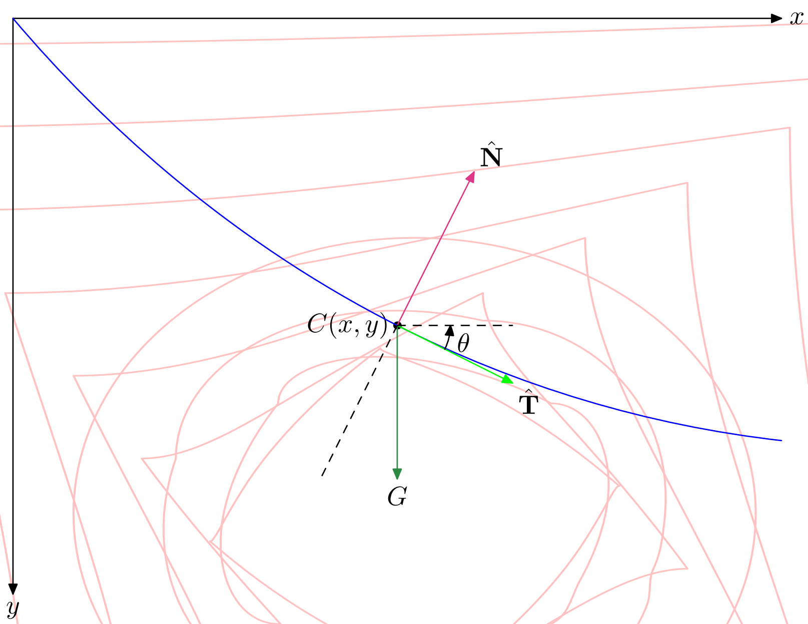
History

In 1696, Johann Bernoulli posed the general problem.

If a wire is bent into an arbitrary curve, which of the infinitely many curves yields the fastest descent?

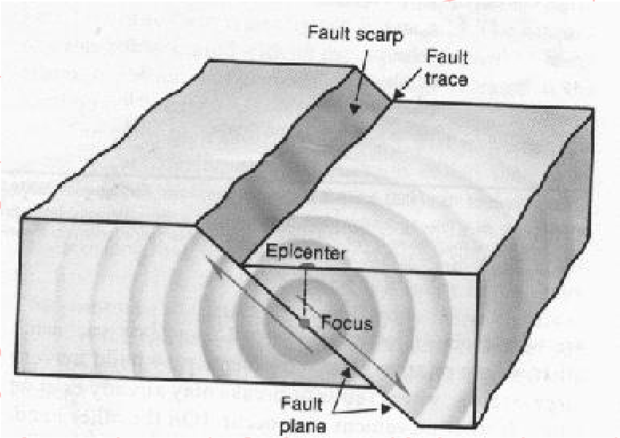
This path is called the brachistochrone (from Greek word *brachistos*, shortest + *chronos*, time.)

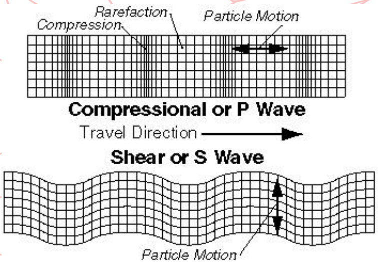
The general Brachistochrone problem is to find the curve joining two points along which a frictionless bead will descend in minimal time



x
y
N̂
C(x, y)
θ
T̂
G

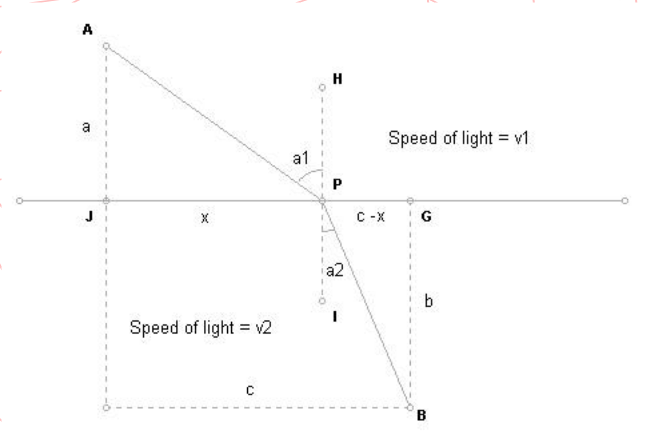
The earth shakes . . .





Fermat's Principle of Least Time

Seismic energy finds the path of shortest travel time (Fermat's Prin-

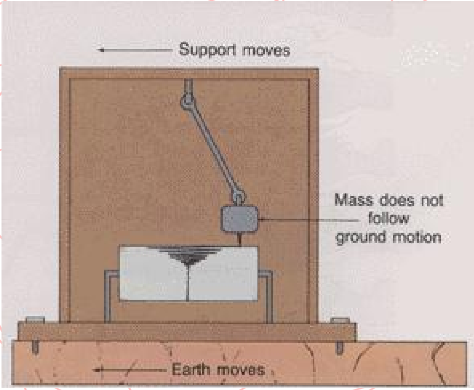


ciple of Least Time.)
Eq NRG attenuates through the earth on curved ray paths.
taking the derivative of time wrt x (to find the minimum):

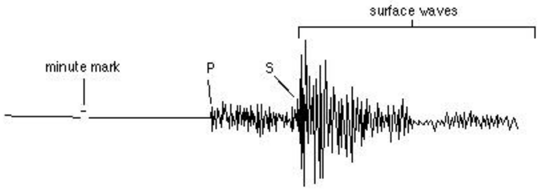
$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2+x^2}} + \frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}v_2$$

Seismograph

- method of observing earthquake:



- seismogram



P waves

- P waves have the fastest arrival time
- the difference between P and S waves gives distance travelled
- Observed data: P wave arrival time, velocity structure of the earth
- Assumed data: uniform composition in each velocity layer
- Can use these values and the shape of the raypath curves with the highest velocities.

Criteria for the fastest curve:

- neglecting curvature, magnitude of friction force less @ steep points on curve (range: 0 @ vert tan to whole weight @ horizontal tan)
- classical Brachistochrone prob says steepness most important initially
  $\mapsto$ suggests steepness more weighted than path length
  $\mapsto$ optimal curve (still needs initial vert tan) will be slightly steeper or below cycloid.
- normal component of acceleration is proportional to sq of speed, expect opposite in frictional model
- starting off steeper forces more curvature for latter portion of path, when there is a greater velocity

Goal

Derive the equation of the curve of fastest time.

Equations

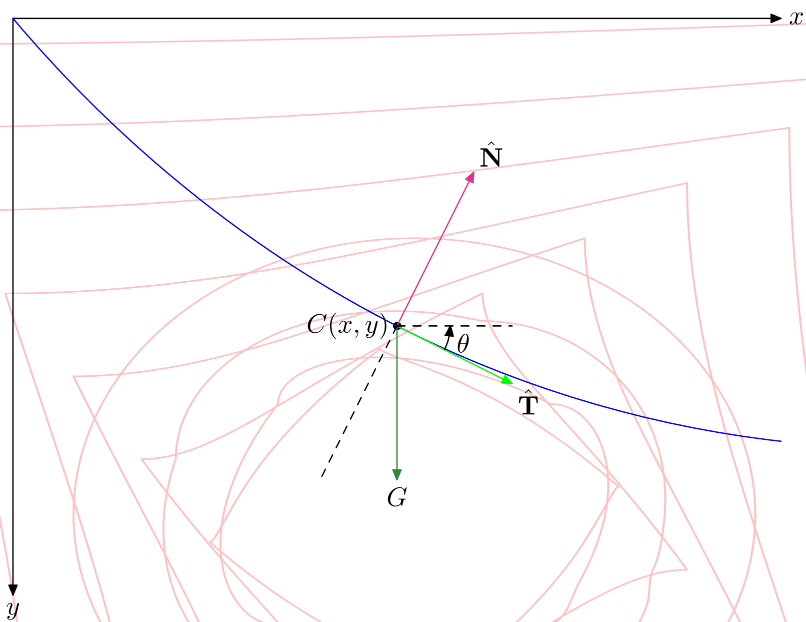
- The unit Normal and unit Tangent vectors:

$$\hat{N} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \qquad \hat{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$$

- The force components in the direction of the unit Tangent vector (along the curve):

$$F_{gravity}\hat{T} = mg\frac{dy}{ds} \qquad F_{friction}\hat{T} = -\mu_k mg\frac{dx}{ds}$$

Interpret subsurface.


x
y
$\hat{\mathbf{N}}$
$C(x, y)$
θ
$\hat{\mathbf{T}}$
G

More equations

Applying the Euler Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

to the time equation:

$$T(x, y, y') = \int_a^b \sqrt{\frac{1 + (y')^2}{2gy(y - \mu_k x}} dx$$

(After what feels like) hundreds of calculations later . . .
and the parametric equations of the cycloid:

$$x_c(\theta) = \rho(\theta - sin(\theta)) \qquad \text{and} \qquad y_c(\theta) = \rho(\theta - cos(\theta))$$

. . . I have the fastest frictional curve, the answer to the Brachistochrone problem:

$$x(\theta) = x_c(\theta) + \rho(\theta - sin(\theta)) \qquad \text{and} \qquad y(\theta) = y_c(\theta) + \rho(\theta - cos(\theta))$$

Pulling it all together

Using:

- model of velocity structure
- Fermat's Principle of Least Time
- the Brachistochrone curve

And a few more term projects such as this, and that for Linear, I will then be able to interpret the subsurface structure of the earth.

Conclusion

THE END