

## Exam #2 Study Page Solutions

1. Simplify each expression.

$$\text{a. } (-1 - \sqrt{3}i)^3 = (-1)^3 - 3(-1)^2(\sqrt{3}i) + 3(-1)(\sqrt{3}i)^2 - (-1)(\sqrt{3}i)^3 = -1 - 3\sqrt{3}i - 3(-1 - 3\sqrt{3}i + 9 - 3\sqrt{3}i) = 8$$

$$\text{b. } \frac{2+3i}{3-4i} = \left(\frac{2+3i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{6+8i+9i-12}{9+16} = \frac{-6+17i}{25} = -\frac{6}{25} + \frac{17}{25}i$$

$$\text{c. } i^{73} = i^{4 \cdot 18 + 1} = i^{4 \cdot 18} i^1 = (i^4)^{18} i = 1^{18} i = i$$

$$\text{d. } \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + 2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}i + \left(\frac{\sqrt{2}}{2}i\right)^2 = \frac{1}{2} + i - \frac{1}{2} = i$$

2. Given  $z = a + bi$  and  $w = c + di$ .

$$\begin{aligned} \overline{z-w} &= \overline{(a+bi) - (c+di)} \\ &= \overline{(a-c) + (b-d)i} \\ &= (a-c) - (b-d)i \\ &= a - c - bi + di \\ &= a - bi - c + di \\ &= (a-bi) - (c-di) \\ &= \overline{z} - \overline{w} \end{aligned}$$

3.

$$\begin{aligned} x^2 - 4x - 6 &= 0 \\ x^2 - 4x + (2)^2 &= 6 + (2)^2 \\ (x-2)^2 &= 10 \\ x-2 &= \pm\sqrt{10} \\ x &= 2 \pm \sqrt{10} \end{aligned}$$

4. Use the quadratic formula to solve:

a. For  $x^2 - 2x = 2$ , need to solve  $x^2 - 2x - 2 = 0$ . Using the quadratic formula gives

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4+8}}{2} \\ &= \frac{2 \pm \sqrt{12}}{2} \\ &= \frac{2(1 \pm \sqrt{3})}{2} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

b. For  $3x^2 + 4 = 10x$ , need to solve  $3x^2 - 10x + 4 = 0$ . Using the quadratic formula gives

$$\begin{aligned}
x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(4)}}{2(3)} \\
&= \frac{10 \pm \sqrt{100 - 48}}{6} \\
&= \frac{10 \pm \sqrt{52}}{6} \\
&= \frac{2(5 \pm \sqrt{13})}{6} \\
&= \frac{5 \pm \sqrt{13}}{3}
\end{aligned}$$

- c. For  $u - \sqrt{u} - 2 = 0$ , let  $x = \sqrt{u}$ . So  $x^2 = u$  and then substituting into the equation gives the quadratic  $x^2 - x - 2 = 0$ . Now we need to solve  $x^2 - x - 2 = 0$  using the quadratic formula.

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{1 \pm \sqrt{1 + 8}}{2} \\
&= \frac{1 \pm 3}{2}
\end{aligned}$$

For each solution above we have to substitute back  $x = \sqrt{u}$  to result in the solutions

$$\begin{aligned}
\sqrt{u} &= 2 \quad \text{or} \quad \sqrt{u} = -1 \\
u &= 4 \quad \text{Square roots can not be negative.}
\end{aligned}$$

- d. For  $m^{2/3} - 2m^{1/3} = 3$ , notice  $(m^{1/3})^2 - 2(m^{1/3}) - 3 = 0$  is also and equivalent equation. Let  $x = m^{1/3}$  and then substituting into the equation gives the quadratic  $x^2 - 2x - 3 = 0$ . Now we need to solve  $x^2 - 2x - 3 = 0$  using the quadratic formula.

$$\begin{aligned}
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 + 12}}{2} \\
&= \frac{2 \pm 4}{2}
\end{aligned}$$

For each solution above we have to substitute back  $x = m^{1/3}$  to result in the solutions

$$\begin{aligned}
m^{1/3} &= 3 \quad \text{or} \quad m^{1/3} = -1 \\
m &= 27 \quad \text{or} \quad m = -1
\end{aligned}$$

5.  $y = x^2 - 2x - 3$   
i.

$$\begin{aligned}
y &= x^2 - 2x - 3 \\
&= x^2 - 2x + (1)^2 - (1)^2 - 3 \\
&= (x - 1)^2 - 4
\end{aligned}$$

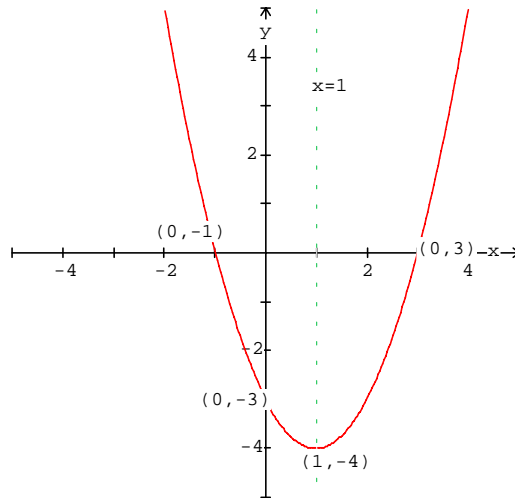
vertex at  $(1, -4)$

ii.  $y = x^2 - 2x - 3 = (x - 3)(x + 1)$

$y$ -intercept:  $(0, -3)$

$x$ -intercepts:  $(-1, 0)$  and  $(3, 0)$

iii.



6.  $y = -x^2 - 4x + 5$

i.

$$\begin{aligned} y &= -x^2 - 4x + 5 \\ &= -(x^2 + 4x + (2)^2) + (2)^2 + 5 \\ &= -(x + 2)^2 + 9 \end{aligned}$$

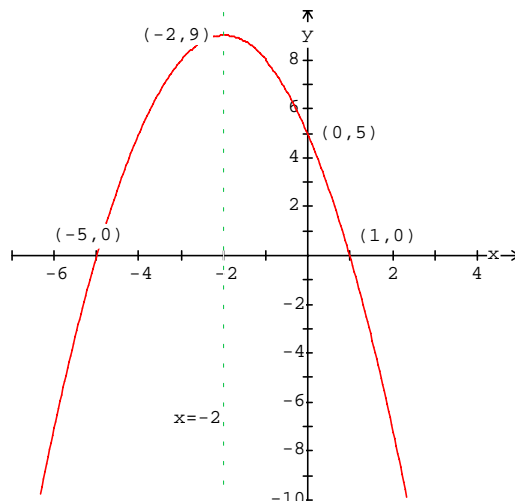
vertex at  $(-2, 9)$

ii.  $y = -x^2 - 4x + 5 = -(x^2 + 4x - 5) = -(x + 5)(x - 1)$

$y$ -intercept:  $(0, 5)$

$x$ -intercepts:  $(-5, 0)$  and  $(1, 0)$

iii.



7.  $f(x) = 2x^2 - 4x - 4$

i.

$$\begin{aligned}y &= 2x^2 - 4x - 4 \\ &= 2(x^2 - 2x + (1)^2) - 2(1)^2 - 4 \\ &= 2(x - 1)^2 - 6\end{aligned}$$

vertex at  $(1, -6)$

ii.  $y = 2x^2 - 4x - 4 = 2(x^2 - 2x - 2)$

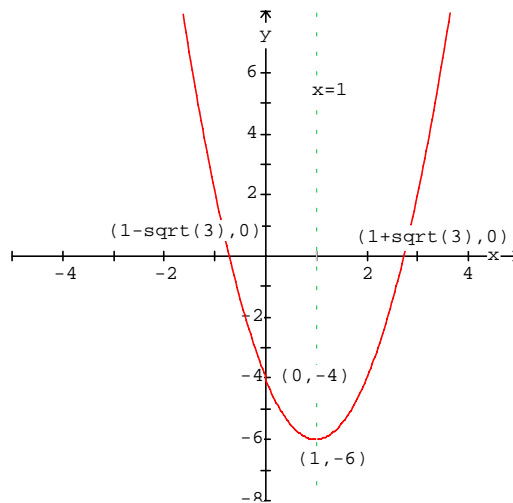
$y$ -intercept:  $(0, -4)$

Since the equation does not factor, use quadratic formula to find the zeros

$$\begin{aligned}x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 + 8}}{2} \\ &= \frac{2 \pm 2\sqrt{3}}{2} \\ &= 1 \pm \sqrt{3}\end{aligned}$$

$x$ -intercepts:  $(1 - \sqrt{3}, 0)$  and  $(1 + \sqrt{3}, 0)$

iii.



8.  $f(x) = -3x^2 + 3x - 6$

i.

$$\begin{aligned}y &= -3x^2 + 3x - 6 \\ &= -3\left(x^2 - x + \left(\frac{1}{2}\right)^2\right) + 3\left(\frac{1}{2}\right)^2 - 6 \\ &= -3\left(x - \frac{1}{2}\right)^2 - \frac{21}{4}\end{aligned}$$

vertex at  $\left(\frac{1}{2}, -\frac{21}{4}\right)$

ii.  $y = -3x^2 + 3x - 6 = -3(x^2 - x + 2)$

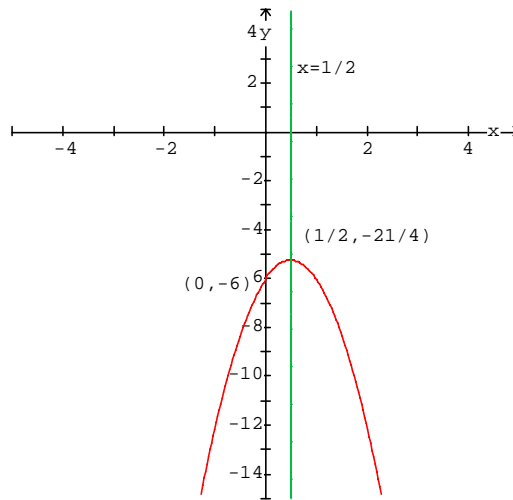
$y$ -intercept:  $(0, -6)$

Since the equation does not factor, use quadratic formula to find the zeros

$$\begin{aligned}
 x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)} \\
 &= \frac{1 \pm \sqrt{1 - 8}}{2} \\
 &= \frac{1 \pm \sqrt{-7}}{2} \\
 &\text{no real roots}
 \end{aligned}$$

$x$  –intercepts: none

iii.



9. Let revenue =  $R$ . Revenue equals the price of a quantity times the number of the quantity sold.

$$\begin{aligned}
 R &= px \\
 &= p(50 - p) \\
 &= -p^2 + 50p \\
 &= -(p^2 - 50p + (25)^2) + 25^2 \\
 &= -(p - 25)^2 + 625
 \end{aligned}$$

With a price of 25 the shopkeeper will attain a maximum revenue of 625.

10. Given the path of the baseball is  $h = -0.0032x^2 + x + 3$ , we know the ball will reach a maximum height at the vertex of the parabola. The  $x$  –coordinate of the vertex is  $x = \frac{-1}{2(-0.0032)} = 156.25$  ft. To find the maximum height put  $x = 156.25$  into the equation for  $h$ .

$$\begin{aligned}
 h &= -0.0032(156.25)^2 + 156.25 + 3 \\
 &= 81.125 \text{ ft}
 \end{aligned}$$

11. For the fence let  $x$  represent the width of the field and  $y$  the length of the field that is parallel to the river. If fencing costs \$12 per foot on the ends and \$18 per foot on the length and the total cost of fencing is \$5400, then  $12(2x) + 18y = 5400$ . The area of the field will be  $A = xy$ . Solving for  $y$  in the equation gives  $y = -\frac{4}{3}x + 300$ . Now we need only find the maximum of

$$A = x\left(-\frac{4}{3}x + 300\right)$$

$$= -\frac{4}{3}x^2 + 300x$$

The maximum will occur at the  $x$ -coordinate of the vertex and the value is

$$x = \frac{-300}{2\left(-\frac{4}{3}\right)} = \frac{225}{2} \text{ ft. To find the length of the field substitute } x \text{ back into our equation}$$

for  $y$  to get  $y = -\frac{4}{3}\left(\frac{225}{2}\right) + 300 = 150$  ft. So the dimensions of the field are 112.5 ft. by 150 ft.

12. To find  $c$  so the  $f(x) = \frac{1}{2}x^2 - cx - 8$  has a minimum value of  $-14$ , recall that the  $x$ -coordinate of the vertex will be  $x = \frac{-(-c)}{2\left(\frac{1}{2}\right)} = c$ . So the point  $(c, -14)$  will be the vertex of the parabola. Substituting into the equation we get

$$f(c) = -14$$

$$\frac{1}{2}(c)^2 - c(c) - 8 = -14$$

$$\frac{1}{2}c^2 - c^2 = -6$$

$$-\frac{1}{2}c^2 = -6$$

$$c^2 = 12$$

$$c = \pm\sqrt{12}$$

$$c = \pm 2\sqrt{3}$$

13. Division Algorithm: When a polynomial  $P(x)$ , called the dividend, is divided by a polynomial  $d(x)$ , called the divisor, the result is another polynomial  $Q(x)$ , called the quotient, and a remainder,  $R(x)$ . Symbolically this is written

$$\frac{P(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)} \text{ where } \deg R(x) < \deg d(x).$$

This may also be written

$$P(x) = d(x) \cdot Q(x) + R(x) \text{ where } \deg R(x) < \deg d(x).$$

14. Divide  $x^4 - 3x^3 + 7x^2 + 21x - 26$  by  $x^2 + x - 2$ .

$$(x^4 - 3x^3 + 7x^2 + 21x - 26)/(x^2 + x - 2) = x^2 - 4x + 13$$

15. *Remainder Theorem*: If a number  $c$  is substituted for  $x$  in the polynomial  $P(x)$ , then the result  $P(c)$  is the remainder that would be obtained by dividing  $P(x)$  by  $x - c$ . That is

$$\text{If } P(x) = (x - c) \cdot Q(x) + R, \text{ then } P(c) = R.$$

16. Find all zeros, both real and complex, of  $p(x) = x^3 - 1$ . Notice  $x = 1$  is a zero of  $p$ , so  $x - 1$  is a factor of  $p(x)$ . Using synthetic or long division gives  $p(x) = (x - 1)(x^2 + x + 1)$ . Now using the quadratic formula gives

$$\begin{aligned}
 x &= \frac{-1 \pm \sqrt{1-4}}{2} \\
 &= \frac{-1 \pm \sqrt{-3}}{2} \\
 &= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i
 \end{aligned}$$

So the roots are  $x = 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

17. **Factor Theorem:** For a polynomial  $P(x)$ , if  $P(c) = 0$ , then  $x - c$  is a factor of  $P(x)$ .

18. **Rational Zero Theorem:** Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where all the coefficients are integers. Consider a rational number denoted by  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime (having not common factor besides 1 and  $-1$ .) If  $\frac{p}{q}$  is a zero of  $P(x)$ , then  $p$  is a factor of  $a_0$  and  $q$  is a factor of  $a_n$ .

19. Possible rational zeros of the polynomial function  $f(x) = 2x^3 + 2x^2 - 16x - 24$  are of the form  $p/q$  where  $p$  is a factor of 24 and  $q$  is a factor of 2. The possible zeros are:

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{3}{2}$$

Using the graph of the polynomial and/or synthetic division shows there is a zero at  $x = 3$  and a zero with multiplicity 2 at  $x = -2$ . Since the zeros of the polynomial are known, so are the factors.

$$\begin{aligned}
 f(x) &= 2x^3 + 2x^2 - 16x - 24 \\
 &= 2(x - 3)(x + 2)^2
 \end{aligned}$$

20. **Fundamental Theorem of Algebra:** Every polynomial of degree  $n$ ,  $n \geq 1$ , with complex coefficients, has at least one zero in the system of complex numbers.

21. If  $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$  is a polynomial with real coefficients, then if  $a + bi$  is a zero, then  $a - bi$  is also a zero of the polynomial.

22. If  $x = -2$  and  $x = 3 + i$  are zeros of a polynomial with rational coefficients, the  $x = 3 - i$  is also a zero. If 3 zeros are known, then three factor of the polynomial are equation also known.

$$\begin{aligned}
 f(x) &= (x + 2)(x - (3 + i))(x - (3 - i)) \\
 &= (x + 2)((x - 3) - i)((x - 3) + i) \\
 &= (x + 2)((x - 3)^2 + 1) \\
 &= (x + 2)(x^2 - 6x + 10) \\
 &= x^3 - 4x^2 - 2x + 20
 \end{aligned}$$

23. If  $x = -1 + i$  is a zero of  $p(x) = x^4 - 4x^2 - 8x - 4$ , then  $x = -1 - i$  is also a zero. Since two zeros are known, so are the two factors  $(x - (-1 + i))$  and  $(x - (-1 - i))$ . One may either use synthetic division with the complex numbers to find the remaining zeros, or multiply the factors and use long division. Multiplying the factors gives

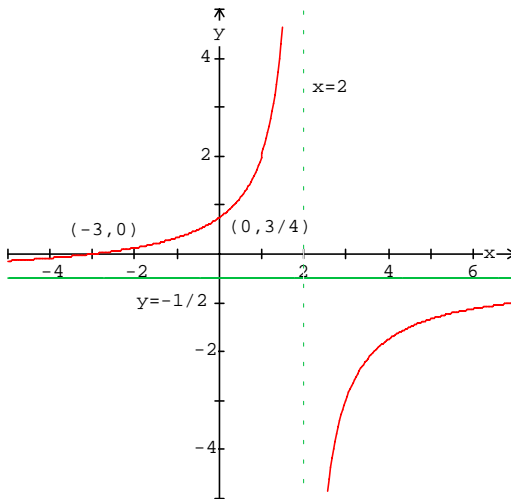
$$\begin{aligned}
 (x - (-1 + i))(x - (-1 - i)) &= ((x + 1) - i)((x + 1) + i) \\
 &= (x + 1)^2 + 1 \\
 &= x^2 + 2x + 2
 \end{aligned}$$

Using long division gives  $(x^4 - 4x^2 - 8x - 4)/(x^2 + 2x + 2) = x^2 - 2x - 2$ . Since  $x^2 - 2x - 2$  does not factor, use the quadratic formula to find zeros at  $x = 1 + \sqrt{3}$ , and  $x = 1 - \sqrt{3}$ . Finally the polynomial factors to

$$p(x) = (x - (-1 + i))(x - (-1 - i))(x - (1 + \sqrt{3}))(x - (1 - \sqrt{3})).$$

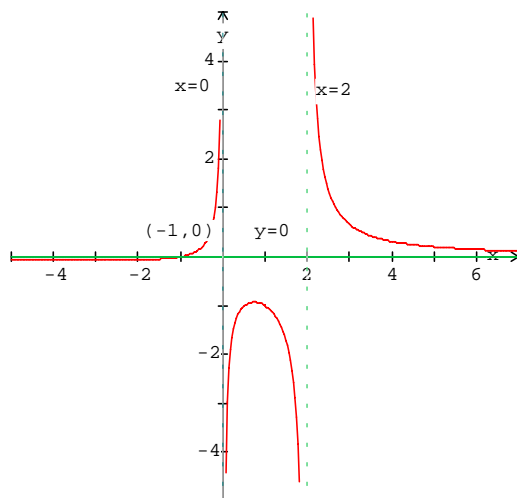
24.  $f(x) = \frac{x+3}{4-2x}$

- a. vertical asymptote at  $x = 2$
- b.  $y$ -intercept at  $y = \frac{3}{4}$   
 $x$ -intercept at  $x = -3$
- c. horizontal asymptote at  $y = -\frac{1}{2}$
- d.



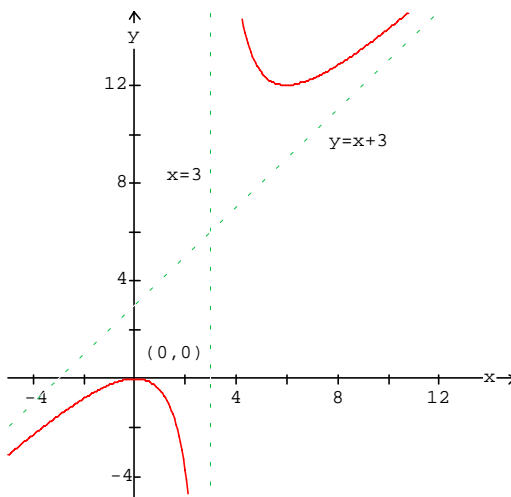
25.  $f(x) = \frac{x+1}{2x^2-4x} = \frac{x+1}{2x(x-2)}$

- a. vertical asymptotes at  $x = 2$  and  $x = 0$
- b.  $x$ -intercept at  $x = -1$   
no  $y$ -intercept
- c. horizontal asymptote at  $y = 0$
- d.



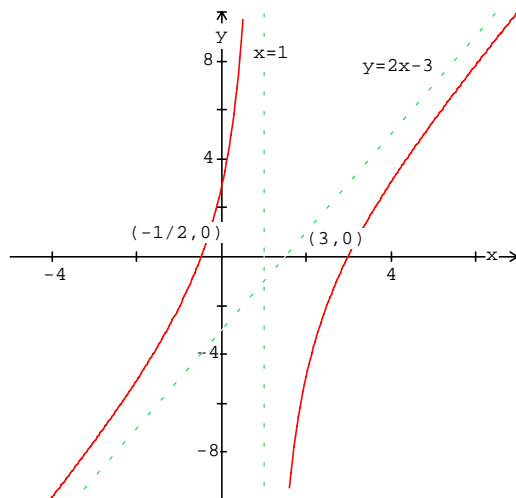
26.  $f(x) = \frac{x^2}{x-3}$

- a. vertical asymptote at  $x = 3$
- b.  $y$ -intercept  $y = 0$   
 $x$ -intercept at  $x = 0$
- c. slant asymptote at  $y = x + 3$
- d.

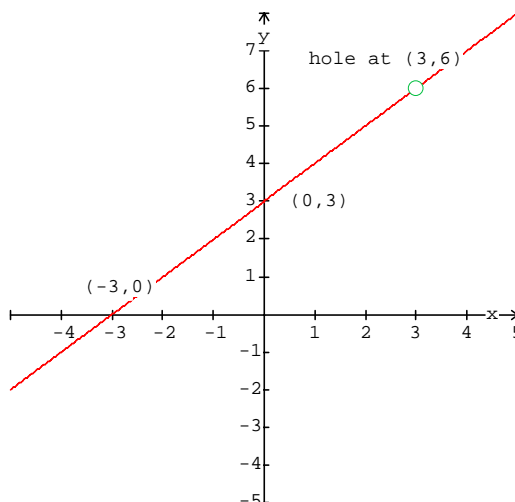


27.  $f(x) = \frac{2x^2 - 5x - 3}{x-1} = \frac{(2x+1)(x-3)}{x-1}$

- a. vertical asymptote at  $x = 1$
- b.  $y$ -intercept  $y = 3$   
 $x$ -intercepts at  $x = 3$  and  $x = -\frac{1}{2}$
- c. slant asymptote at  $y = 2x - 3$
- d.



28.  $f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$  provided  $x \neq 3$ .



29.

$$2x^2 - 4x > 5$$

$$2x^2 - 4x - 5 > 0$$

Using quadratic formula to find the roots to  $2x^2 - 4x - 5$  yields  $x = 1 + \frac{1}{2}\sqrt{14}$ , and  $x = 1 - \frac{1}{2}\sqrt{14}$ . So the solution to the inequality is

$$\left(-\infty, 1 - \frac{1}{2}\sqrt{14}\right) \cup \left(1 + \frac{1}{2}\sqrt{14}, \infty\right).$$

30.

$$2x^3 - 5x^2 - x + 6 \leq 0$$

Use the Rational Zero Theorem, your graphing calculator, and synthetic division to factor the left-hand side of the inequality to get  $2x^3 - 5x^2 - x + 6 = (x - 2)(2x - 3)(x + 1)$ . Now from the graph we find the solution to the inequality is

$$(-\infty, -1] \cup \left[\frac{3}{2}, 2\right]$$

31.

$$\frac{2x+3}{x-1} \leq 0$$

The critical values of the inequality are  $x = \frac{-3}{2}$  and  $x = 1$ . This breaks the number line into 3 pieces  $(-\infty, \frac{-3}{2})$ ,  $(\frac{-3}{2}, 1)$ , and  $(1, \infty)$ . For a test value in the interval  $(-\infty, \frac{-3}{2})$ , both the numerator and the denominator are negative, so the ratio is positive. For a test value in the interval  $(\frac{-3}{2}, 1)$ , the numerator is positive and the denominator is negative, so the ratio is negative. For a test value in the interval  $(1, \infty)$ , both the numerator and the denominator are positive, so the ratio is positive. So the solution to the inequality is

$$\left[ \frac{-3}{2}, 1 \right)$$

32.

$$\frac{x+1}{x-3} \leq \frac{x}{x+1}$$

$$\frac{x+1}{x-3} - \frac{x}{x+1} \leq 0$$

$$\frac{(x+1)(x+1) - x(x-3)}{(x-3)(x+1)} \leq 0$$

$$\frac{5x+1}{(x-3)(x+1)} \leq 0$$

The critical values of the inequality are  $x = -1$ ,  $x = \frac{-1}{5}$  and  $x = 3$ . This breaks the number line into 4 pieces  $(-\infty, -1)$ ,  $(-1, -\frac{1}{5})$ ,  $(-\frac{1}{5}, 3)$  and  $(3, \infty)$ . For a test value in the interval  $(-\infty, -1)$ , the numerator and both factors in the denominator are negative, so the ratio is negative. For a test value in the interval  $(-1, -\frac{1}{5})$ , the numerator is negative and one factor in the denominator is negative while the other is positive, so the ratio is positive. For a test value in the interval  $(-\frac{1}{5}, 3)$ , the numerator is positive and one factor in the denominator is negative while the other is positive, so the ratio is negative. For a test value in the interval  $(3, \infty)$ , the numerator and both factors in the denominator are positive, so the ratio is positive. So the solution to the inequality is

$$(-\infty, -1) \cup \left[ -\frac{1}{5}, 3 \right)$$