College of the Redwoods
Mathematics Department

Math 30—College Algebra

Exam #3—Chapter 4

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Last Revision Date: April 11, 2000 Version 1.00

Multiple Choice Questions

Directions: *In each of the following exercises, select the "best" answer and darken the corresponding oval on your scantron sheet.*

1. Which of the following best defines the inverse of $f(x) = 3x - 5$?

(a) $f^{-1}(x) = (x-3)/5$

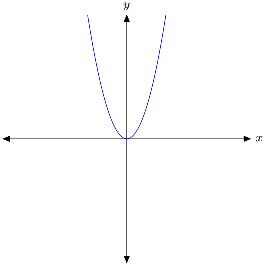
(b) $f^{-1}(x) = (x-5)/3$

(c) $f^{-1}(x) = 1/(3x-5)$

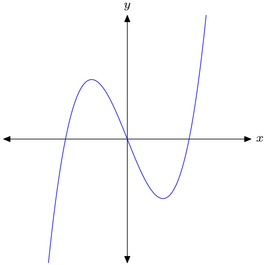
(d) $f^{-1}(x) = (x+3)/5$

(e) $f^{-1}(x) = (x+5)/3$

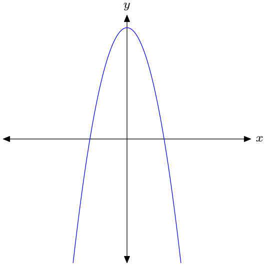
2. Which of the following functions have inverses that are functions?



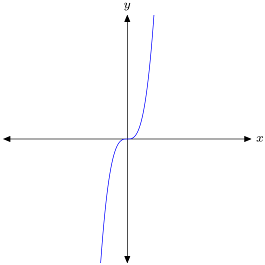
(a)



(b)



(c)



(d)

(e) None of the above

3. If \$1,000 is invested in an account that pays 6% per year, compounded quarterly, how much will be in the account at the end of 10 years? Assume no additional deposits or withdrawals.

(a) \$1,214.79 (b) \$1,541.45 (c) \$1,814.01
(d) \$1,745.38 (e) None of the above

4. How much should be invested in an account that pays 6% per year, compounded continuously, so that there will be \$10,000 in the account ten years from now?

(a) \$5,488.11 (b) \$6,122.56 (c) \$4,923.78
(d) \$5,012.66 (e) None of the above

5. Suppose that $a > 0$. The graph of $f(x) = b + e^{-ax}$ has a horizontal asymptote. What is the equation of this horizontal asymptote?

(a) $y = b - 1$ (b) $y = b$ (c) $y = b + 1$

(d) $y = \frac{b}{a}$ (e) $y = \frac{a}{b}$

6. The expression

$$\frac{\ln a}{\ln b}$$

is equivalent to which of the following expressions?

(a) $\ln \frac{a}{b}$ (b) $\ln a - \ln b$ (c) $\ln a + \ln b$

(d) $\ln a^b$ (e) None of these

7. $e^{\ln x + \ln y}$ equals

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $x + y$

(d) xy (e) None of these

8. $\log_a x$ is identical to

(a) $\dfrac{\ln a}{\ln x}$ (b) $\log_x a$ (c) $\dfrac{\log x}{\log a}$

(d) $\log x^a$ (e) None of these

9. If $x > 0$, then $10^{-2\log(1/x)}$ equals

(a) x^2 (b) $\dfrac{1}{x^2}$ (c) $-2x$

(d) $-x^2$ (e) $-\dfrac{2}{x}$

10. Assume that $r > 0$. The graph of the logistic equation

$$y = \frac{a}{1 + ce^{-rt}}$$

has a horizontal asymptote. What is the equation of this asymptote?

(a) $y = c$ (b) $y = a$ (c) $y = \dfrac{a}{1+c}$

(d) $y = \dfrac{a}{c}$ (e) $y = \dfrac{c}{a}$

Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Solve each of the following equations for x. You must show all of your work to receive full credit. Exact answers only, please. No decimal approximations.

(a) $2^x = 3^{2x+5}$

(b) $e^x - 3e^{-x} = 2$

(c) $\log_2 x + \log_2(x-3) = 2$

Exercise 2. Assume that a colony of bacteria grows according to the model

$$P = P_0 e^{rt}.$$

(a) If the initial bacteria population is 1,000, find P_0.

(b) After 10 days, the bacteria population triples. Use this information to find the growth rate r.

(c) What will be the bacteria population after 15 days? Round your answer to the nearest bacteria.

Exercise 3. Radium-226 has a half-life of 1,620 years. Assume that the radium decays according to the model

$$N = N_0 e^{-kt}.$$

(a) Use the half-life of radium-226 to determine the decay rate k.

(b) How long will it take 2,500 mg of radium-226 to decay to 50 mg? Round your answer to the nearest tenth of a year.

Solutions to Quizzes

Solution to Question 1: When you evaluate $f(x) = 3x - 5$, you begin with a number x, multiply by 3, then subtract 5. The inverse is to start with a number x, add 5, then divide by 3. Consequently, $f^{-1}(x) = (x + 5)/3$. Alternatively, one can exchange x and y in the formula $y = 3x - 5$, then solve for y.

$$\begin{aligned} x &= 3y - 5 \\ 3y &= x + 5 \\ y &= \frac{x + 5}{3} \end{aligned}$$

Therefore,

$$f^{-1}(x) = \frac{x + 5}{3}.$$

□

Solution to Question 2: In Figures (a), (b), and (c), in each case it is possible to cut the graph in more than one place with a horizontal line. Consequently, the function in Figures (a), (b), and (c) are not one-to-one functions and their inverses do not exist (are not functions). However, no horizontal line can cut the graph in Figure (d) more than once (the function in this case is one-to-one). Consequently, when the graph is reflected across the line $y = x$, no vertical line will cut the reflection more than once. Thus, this inverse exists and is a function. □

Solution to Question 3: Use the "discrete" formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A = 1000\left(1 + \frac{0.06}{4}\right)^{(4)(10)}$$
$$A \approx \$1,814.01$$

□

Solution to Question 4: Use the "continuous" formula.

$$A = Pe^{rt}$$

$$10,000 = Pe^{(0.06)(10)}$$

$$P = \frac{10,000}{e^{(0.06)(10)}}$$

$$P \approx \$5,488.11$$

□

Solution to Question 5: If $a > 0$, then $e^{-ax} \to 0$ as $t \to +\infty$. Thus, $f(x) = b + e^{-ax} \to b$ as $x \to +\infty$ and the graph has a horizontal asymptote $y = b$. □

Solution to Question 6: It is a common misconception that $\ln a / \ln b$ is identical to either $\ln(a/b)$ or $\ln a - \ln b$. This is not the case, as is easily checked on your calculator. The correct answer to this question is (e), "None of these." □

Solution to Question 7: The change of base rule

$$\log_b x = \frac{\log_b x}{\log_b a},$$

when applied to $\log_a x$ in terms of the base ten logarithm, provides

$$\log_a x = \frac{\log x}{\log a}.$$

□

Solution to Question 8: The change of base rule,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

allows us to write, using base ten logarithms,

$$\log_a x = \frac{\log x}{\log a}.$$

□

Solution to Question 9: If $x > 0$, then

$$\begin{aligned} 10^{-2\log(1/x)} &= 10^{\log(1/x)^{-2}}, \\ &= 10^{\log x^2}, \\ &= x^2. \end{aligned}$$

$\square$

Solution to Question 10: If $r > 0$, then $e^{-rt} \to 0$ as $t \to +\infty$. Thus,

$$\lim_{t\to+\infty} \frac{a}{1+ce^{-rt}} = \frac{a}{1+0} = a.$$

Therefore, the graph has a horizontal asymptote $y = a$. □

Solutions to Exercises

Exercise 1(a) Take the natural logarithm of each side and isolate x.

$$\begin{aligned}
2^x &= 3^{2x+5} \\
\ln 2^x &= \ln 3^{2x+5} \\
x \ln 2 &= (2x+5) \ln 3 \\
x \ln 2 &= 2x \ln 3 + 5 \ln 3 \\
x \ln 2 - 2x \ln 3 &= 5 \ln 3 \\
x(\ln 2 - 2 \ln 3) &= 5 \ln 3 \\
x &= \frac{5 \ln 3}{\ln 2 - 2 \ln 3}
\end{aligned}$$

□

Exercise 1(b) First, invert.

$$e^x - 3e^{-x} = 2$$
$$e^x - \frac{3}{e^x} = 2$$

Multiply both sides by e^x.

$$e^{2x} - 3 = 2e^x$$
$$e^{2x} - 2e^x - 3 = 0$$

Let $u = e^x$. Then $u^2 = e^{2x}$ and

$$u^2 - 2u - 3 = 0,$$
$$(u-3)(u+1) = 0,$$
$$u = 3, -1.$$

But $u = e^x$, so

$$e^x = 3 \qquad \text{or} \qquad e^x = -1.$$

However, e^x is always positive (the graph of e^x is always above the x-axis). Consequently, $e^x = -1$ has no solutions and the only solution is provided by

$$\begin{aligned} e^x &= 3, \\ x &= \ln 3. \end{aligned}$$

□

Exercise 1(c)

$$
\begin{aligned}
\log_2 x + \log_2(x-3) &= 2 \\
\log_2 x(x-3) &= 2 \\
x(x-3) &= 2^2 \\
x^2 - 3x &= 4 \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x &= 4, -1
\end{aligned}
$$

However, inserting $x = -1$ in the original equation gives

$$\log_2(-1) + \log_2(-1-3) = 2.$$

Because you cannot take the logarithm of a negative number, $x = -1$ cannot be a solution. Hence, the only solution is $x = 4$ (we leave it to you to check this solution in the original equation). □

Exercise 2(a) If the initial population is 1,000, then

$$\begin{aligned} P(0) &= 1,000, \\ P_0 e^{r(0)} &= 1,000, \\ P_0 e^{0} &= 1,000, \\ P_0 &= 1,000. \end{aligned}$$

This last result is true because $e^0 = 1$. We now have

$$P = 1,000e^{rt}.$$

□

Exercise 2(b) After 10 days the population triples. Thus,

$$\begin{aligned} P(10) &= 3,000, \\ 1,000e^{r(10)} &= 3,000, \\ e^{10r} &= 3, \\ 10r &= \ln 3, \\ r &= \frac{1}{10} \ln 3. \end{aligned}$$

We now have

$$P = 1,000e^{((1/10)\ln 3)t}.$$

□

Exercise 2(c) After 15 days,

$$P(15) = 1,000e^{(1/10)\ln 3)(15)},$$
$$P(15) \approx 5196.$$

□

Exercise 3(a) Half of the initial substance will remain after 1,620 years.

$$\begin{aligned} \frac{1}{2}N_0 &= N_0 e^{-k(1{,}620)} \\ \frac{1}{2} &= e^{-1{,}620k} \\ -1,620k &= \ln(1/2) \\ k &= -\frac{1}{1,620}\ln(1/2) \end{aligned}$$

We now have

$$N = N_0 e^{((1/1{,}620)\ln(1/2))t}.$$

□

Exercise 3(b) Let $N_0 = 2,500$ mg and $N = 50$ mg and solve for t.

$$
\begin{aligned}
50 &= 2,500e^{((1/1620)\ln(1/2))t} \\
\frac{1}{50} &= e^{((1/1620)\ln(1/2))t} \\
\left(\left(\frac{1}{1,620}\right)\ln\frac{1}{2}\right)t &= \ln\frac{1}{50} \\
t &= \frac{\ln(1/50)}{(1/1,620)\ln(1/2)} \\
t &= \frac{1,620\ln(1/50)}{\ln(1/2)} \\
t &\approx 9,143.0\,\text{years}
\end{aligned}
$$

□