College of the Redwoods
Mathematics Department
Math 30–College Algebra

Exam #5
Sequences and Series

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Last Revision Date: May 10, 2001 Version 1.00

Multiple Choice Questions

Directions: ***In each of the following exercises, select the "best" answer and darken the corresponding oval on your scantron sheet.***

1. Find the n$^{\text{th}}$ term of the sequence

$$1,\ -8,\ 27, -64,\ \ldots.$$

(a) $a_n = n^3$ **(b)** $a_n = (-1)^n n^3$ **(c)** $a_n = (-1)^{n+1} n^3$
(d) $a_n = (-1)^n n^2$ **(e)** $a_n = (-1)^{n+1} n^2$

2. Find the 4$^{\text{th}}$ term of the sequence

$$a_n = nx^{n-1}.$$

(a) $5x^4$ **(b)** $4x^4$ **(c)** $4x^3$ **(d)** $5x^5$ **(e)** $4x^2$

3. Find the 4$^{\text{th}}$ term of the recursive sequence

$$a_n = 3a_{n-1} + 2$$

where $a_1 = 4$.

(a) 50 **(b) 152** **(c) 458** **(d) 14** **(e) None of these**

4. Find the 5$^{\text{th}}$ term of the recursive sequence

$$a_{n+2} = 2a_{n+1} + a_n$$

where $a_1 = 1$ and $a_2 = 1$.

(a) 5 **(b) 11** **(c) 15** **(d) 17** **(e) None of these**

5. Compute the sum

$$\sum_{k=1}^{3} \frac{k}{k+1}.$$

(a) $\frac{6}{9}$ **(b)** $\frac{7}{11}$ **(c)** $\frac{23}{12}$ **(d)** $\frac{19}{12}$ **(e) None of these**

6. Find the nth term of the *arithmetic* sequence

$$-2,\ 1,\ 4,\ 7,\ 10,\ \ldots.$$

(a) $a_n = -2(3)^n$ **(b)** $a_n = -2 + 3n$ **(c)** $a_n = -3 + 2n$
(d) $a_n = -5 + 3n$ **(e)** $a_n = 3(-2)^n$

7. Find the two-hundredth term, a_{200}, of the sequence

$$1,\ 5,\ 9,\ 13,\ \ldots.$$

(a) 797 **(b) 801** **(c) 805** **(d) 809** **(e) None of these**

8. The second term of an *arithmetic* sequence is $a_2 = 13$ and the seventh term is $a_7 = 68$. Find the n^{th} term, a_n, of the sequence.

(a) $a_n = 2 + 8(n-1)$ **(b)** $a_n = 3 + 4(n-1)$ **(c)** $a_n = 1 + 7(n-1)$
(d) $a_n = 2 + 11(n-1)$ **(e) None of these**

9. Calculate the sum

$$\sum_{k=1}^{100} 3k + 4.$$

(a) 14,550 **(b) 17,800** **(c) 13,450** **(d) 12,500** **(e) None of these**

10. Calculate the sum of the *arithmetic* series

$$3 + 5 + 7 + \cdots + 101.$$

(a) 2598 **(b) 2600** **(c) 2602** **(d) 2604** **(e) None of these**

11. Find the common ratio of the *geometric* sequence

$$8,\ 12,\ 18,\ \ldots$$

(a) 3/4 **(b) 2/3** **(c) 3/2** **(d) 1/2** **(e) None of these**

12. The $\mathbf{n^{th}}$ term of the *geometric* sequence

$$2,\ -6,\ 18,\ -54,\ \ldots$$

is

(a) $a_n = 3(-2)^n$ **(b)** $a_n = 2(-3)^n$ **(c)** $a_n = 2(-3)^{n-1}$

(d) $a_n = 2(-2/3)^{n-1}$ **(e) None of these**

13. Find the sum of the first 10 terms of the *geometric* sequence

$$1,\ 3,\ 9,\ 27,\ \ldots$$

(a) 29,524 **(b) 9,841** **(c) 88,573** **(d) 31,534** **(e) None of these**

14. Compute the sum

$$1 - x + x^2 - x^3 + x^4 - x^5.$$

(a) $\dfrac{1-x^5}{1+x}$ **(b)** $\dfrac{1+x^5}{1+x}$ **(c)** $\dfrac{1+x^6}{1+x}$

(d) $\dfrac{1-x^6}{1+x}$ **(e)** $\dfrac{1}{1-a}$

15. Find the sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{3}{2^n}.$$

(a) 3 **(b) 4** **(c) 6** **(d) 3/2** **(e) None of these**

16. Find the sum of the infinite series

$$1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots$$

given $|x-2| < 1$**.**

(a) $\dfrac{1}{x-2}$ **(b)** $\dfrac{1}{2-x}$ **(c)** $\dfrac{1}{3-x}$ **(d)** $\dfrac{1}{1-x}$ **(e) None of these**

17. A six sided die contains on each of its faces one of the numbers 1, 2, 3, 4, 5, or 6. Once a number is selected for the face, it may not be reused. Thus, each face of the die has a unique number, selected from the numbers 1 through 6. A fair coin has two sides, heads

or tails. If the die is rolled and the coin is tossed, one possible outcome is 6T, meaning you rolled a six and tossed a tail. How many different outcomes are possible?

(a) **8** (b) **10** (c) **12** (d) **32** (e) **None of these**

18. Mr. Marks likes to give 10 question "true-false" quizzes during the week. This means that the quiz has ten questions, each of which can be marked "true" or "false." Out of boredom, Mr. Marks decides to make answer keys for the exam by randomly selecting "true" or "false" on each of the ten questions. With this strategy, how many answer keys are possible?

(a) **20** (b) **1024** (c) **200** (d) **2000** (e) **None of these**

19. Jane has five books that she treasures: "A Map of the World," "The Shipping News," "Cold Mountain," "Snow Falling on Cedars," and "Harry Potter and the Chamber of Secrets." In how many different ways can she arrange these books on her bookshelf?

(a) **10** (b) **32** (c) **120** (d) **5** (e) **None of these**

20. Compute ${}_nP_{n-1}$.

(a) n (b) $n(n-1)$ (c) $n(n+1)$ (d) $n(n-1)/2$ (e) **None of these**

21. Plot 10 points in a plane, no three of which are collinear. How many different lines can be drawn, if each line must pass through 2 of the 10 points?

(a) **45** (b) **20** (c) **200** (d) **90** (e) **None of these**

22. Ten dogs compete for 3 ribbons: blue for first place, red for second place, and white for third place. If the judge selects three of ten dogs at random, in how many different ways can the ribbons be awarded with this strategy?

(a) ${}_7P_3$ (b) ${}_{10}C_3$ (c) ${}_7C_3$ (d) ${}_{10}P_3$ (e) **None of these**

23. Compute ${}_nC_2$.

(a) $n(n+1)$ (b) $n(n-1)$ (c) $n(n-1)(n-2)/6$

(d) $n(n-1)/2$ (e) **None of these**

24. Solve for n.

$$\binom{n+1}{3} = 2 \cdot \binom{n}{2}$$

(a) **3** (b) **4** (c) **5** (d) **6** (e) **None of these**

25. Calculate the next row of Pascal's Triangle.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

(a) 1 5 7 7 5 1 (b) 1 5 10 10 5 1 (c) 1 6 10 10 6 1

(d) 1 6 7 7 6 1 (e) **None of these**

26. Use the binomial theorem to expand

$$(x-2y)^3.$$

(a) $x^3 - 8y^3$ (b) $x^3 - 6xy - 8y^3$

(c) $x^3 - 6x^2y + 12xy^2 - 8y^3$ (d) $x^3 - 3x^2y + 3xy^2 - 8y^3$

(e) **None of these**

27. Find the 11$^{\text{th}}$ term of

$$(x-2y)^{10}.$$

(a) $180x^8y^2$ **(b)** $-20xy^4$ **(c)** $60x^8y^2$ **(d)** $-20x^9y$ **(e) None of these**

28. One of the solutions of

$$\sum_{k=0}^{4}\binom{4}{k}(x)^{4-k}(3)^k=1$$

is

(a) -2 **(b) 3** **(c)** -6 **(d) 1** **(e) None of these**

29. Simplify

$$\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}+\binom{7}{4}+\binom{7}{5}+\binom{7}{6}+\binom{7}{7}.$$

(a) 32 **(b) 64** **(c) 128** **(d) 256** **(e) None of these**

Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Use mathematical induction to show that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Solutions to Multiple Choice Questions

Solution to Question 1: Note that each term is the cube of a natural number and that the signs alternated. The formula

$$a_n = (-1)^{n+1} n^3$$

seems the appropriate choice, but let's check.

$$\begin{aligned} a_1 &= (-1)^{1+1}(1)^3 = (1)(1) = 1 \\ a_2 &= (-1)^{2+1}(2)^3 = (-1)(8) = -8 \\ a_3 &= (-1)^{3+1}(3)^3 = (1)(27) = 27 \\ a_4 &= (-1)^{4+1}(4)^3 = (-1)(64) = -64 \end{aligned}$$

Thus, all is well and the solution is

$$a_n = (-1)^{n+1} n^3.$$

□

Solution to Question 2: To find the 4^{th} term, substitute $n = 4$ in

$$\begin{aligned} a_n &= nx^{n-1} \\ a_4 &= 4x^{4-1} \\ a_4 &= 4x^3. \end{aligned}$$

□

Solution to Question 3: The recursive sequence

$$a_n = 3a_{n-1} + 2,$$

together with initial condition $a_1 = 4$, gives

$$\begin{aligned} a_n &= 3a_1 + 2 = 3(4) + 2 = 14 \\ a_3 &= 3a_2 + 2 = 3(14) + 2 = 44 \\ a_4 &= 3a_3 + 2 = 3(44) + 2 = 134. \end{aligned}$$

□

Solution to Question 4: The recursive definition

$$a_{n+2} = 2a_{n+1} + a_n,$$

together with initial conditions $a_1 = 1$ and $a_2 = 1$, lead to

$$\begin{aligned} a_3 &= 2a_2 + a_1 = 2(1) + 1 = 3 \\ a_4 &= 2a_3 + a_2 = 2(3) + 1 = 7 \\ a_5 &= 2a_4 + a_3 = 2(7) + 3 = 17. \end{aligned}$$

□

Solution to Question 5: Summing,

$$\begin{aligned} \sum_{k=1}^{3} \frac{k}{k+1} &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \\ &= \frac{6}{12} + \frac{8}{12} + \frac{9}{12} \\ &= \frac{23}{12} \end{aligned}$$

□

Solution to Question 6: The sequence

$$-2,\ 1,\ 4,\ 7,\ 10, \ldots$$

is arithmetic, with common difference $d = 3$. Thus, the n^{th} term is

$$\begin{aligned} a_n &= a + (n-1)d \\ a_n &= -2 + (n-1)3 \\ a_n &= -2 + 3n - 3 \\ a_n &= 3n - 5. \end{aligned}$$

□

Solution to Question 7: The sequence

$$1,\ 5,\ 9,\ 13, \ldots$$

is arithmetic, with common ratio $d = 4$, so

$$\begin{aligned} a_n &= a + (n-1)d \\ a_n &= 1 + (n-1)4. \end{aligned}$$

Thus,

$$a_{200} = 1 + (200-1)4 = 797$$

□

Solution to Question 8: The n^{th} term of an arithmetic sequence is

$$a_n = a + (n-1)d.$$

Thus, $a_2 = 13$ gives

$$\begin{aligned} a_2 &= a + (2-1)d \\ 13 &= a + d. \end{aligned} \tag{1}$$

Also, $a_7 = 68$ gives

$$\begin{aligned} a_7 &= a + (7-1)d \\ 68 &= a + 6d \end{aligned} \tag{2}$$

Solve equation (1) for a:

$$a = 13 - d \tag{3}$$

Sub (3) in (2):

$$\begin{aligned} 68 &= (13-d) + 6d \\ 68 &= 13 + 5d \\ 55 &= 5d \\ d &= 11 \end{aligned}$$

Sub $d = 11$ in (3):

$$\begin{aligned} a &= 13 - 11 \\ a &= 2. \end{aligned}$$

Therefore,

$$\begin{aligned} a_n &= a + (n-1)d \\ a_n &= 2 + (n-1)11 \end{aligned}$$

□

Solution to Question 9: The sum

$$S = \sum_{k=1}^{100} (3k+4) = 7 + 10 + 13 + \cdots + 304$$

is arithmetic, so we can use

$$S = \frac{n(a_1 + a_n)}{2}$$
$$S = \frac{100(7 + 304)}{2}$$
$$S = 15,550$$

□

Solution to Question 10: The trick is to determine the number of terms. The n^{th} term of an arithmetic sequence is

$$a_n = a + (n-1)d.$$

Because

$$S = 3 + 5 + 7 + \cdots + 101,$$

we have $a = 3$ and $d = 2$, so

$$a_n = 3 + (n-1)2$$
$$a_n = 2 + 2n - 2$$
$$a_n = 1 + 2n.$$

Use $a_n = 101$ to determine the number of terms.

$$a_n = 101$$
$$1 + 2n = 101$$
$$2n = 100$$
$$n = 50$$

Thus, there are 50 terms. We can now use

$$S = \frac{n(a_1 + a_n)}{2}$$
$$S = \frac{50(a_1 + a_50)}{2}$$
$$S = \frac{50(3 + 101)}{2}$$
$$S = 2600.$$

□

Solution to Question 11: In the series

$$8,\ 12,\ 18, \ldots$$

note that

$$\frac{12}{8} = \frac{3}{2}$$

and

$$\frac{18}{12} = \frac{3}{2},$$

so the common ratio is 3/2. □

Solution to Question 12: In the series,

$$2,\ -6,\ 18,\ -54, \ldots$$

The common ratio is

$$-3 = \frac{-6}{2} = \frac{18}{-6} = \frac{-54}{18} = \cdots$$

The n^{th} term is

$$a_n = ar^{n-1}$$
$$a_n = 2(-3)^{n-1}.$$

□

Solution to Question 13: The geometric sequence

$$1,\ 3,\ 9,\ 27, \ldots$$

has common ratio 3, so the sum of the 1st terms is

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_{10} = \frac{(1)(1-3^{10})}{1-3}$$
$$S_{10} = 29,524$$

□

Solution to Question 14: The series

$$1 - x + x^2 - x^3 + x^4 - x^5$$

is geometric with common ration $r = -x$. Thus, the sum of these 6 terms is

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_6 = \frac{1(10(-x)^6)}{1-(-x)}$$
$$S_6 = \frac{1-x^6}{1+x}.$$

□

Solution to Question 15: The infinite series

$$S = \sum_{n=0}^{\infty} \frac{3}{2^n} = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$$

is geometric with common ration $r = 1/2$. Because $-1 < r < 1$, the series converges and

$$S = \frac{a}{1-r}$$
$$S = \frac{3}{1-1/2}$$
$$S = 6.$$

□

Solution to Question 16: The infinite series

$$S = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots$$

is geometric with common ratio $r = x - 2$. Since $|x-2| < 1$, the series converges and

$$S = \frac{a}{1-r}$$
$$S = \frac{1}{1-(x-2)}$$
$$S = \frac{1}{3-x}$$

□

Solution to Question 17: There are 6 outcomes for the die, 2 for the coin. The multiplication principle gives a total of

$$6 \cdot 2 = 12$$

possible outcomes. □

Solution to Question 18: Because each of 10 questions has 2 possible answers, the multiplication principle gives

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024$$

possible answers. □

Solution to Question 19: The 5 books can be arranged in

$${}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120$$

different ways. □

Solution to Question 20: Compute

$$\begin{aligned} {}_nP_{n-1} &= \frac{n!}{(n-(n-1))!} \\ &= \frac{n!}{(n-n+1)!} \\ &= \frac{n!}{1!} \\ &= n! \end{aligned}$$

□

Solution to Question 21: Because 2 points determine a line, and no 3 points are collinear, the number of lines equals the number of ways that we can choose 2 points out of 10.

$$\begin{aligned} {}_{10}C_2 &= \frac{10!}{2!(10-2)!} \\ &= \frac{10 \cdot 9 \cdot 8!}{2!8!} \\ &= \frac{10 \cdot 9}{2} \\ &= 5 \cdot 9 \\ &= 45 \end{aligned}$$

□

Solution to Question 22: Order matters (first, second, and third place), so the number of ways we can order 10 dogs taken 2 at a time is

$$\begin{aligned} {}_{10}P_3 &= \frac{10!}{(10-3)!} \\ &= \frac{10!}{7!} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \\ &= 10 \cdot 9 \cdot 8 \\ &= 720. \end{aligned}$$

□

Solution to Question 23: Compute

$$
\begin{aligned}
{}_nC_2 &= \frac{n!}{2!(n-2)!} \\
&= \frac{n(n-1)(n-2)!}{2(n-2)!} \\
&= \frac{n(n-1)}{2}.
\end{aligned}
$$

□

Solution to Question 24: Solve for n.

$$
\begin{aligned}
\binom{n+1}{3} &= 2\binom{n}{2} \\
\frac{(n+1)!}{3!((n+1)-3)!} &= 2 \cdot \frac{n!}{2!(n-2)!} \\
\frac{(n+1)!}{3 \cdot 2!(n-2)!} &= \frac{2n!}{2!(n-2)!}
\end{aligned}
$$

Multiply both sides by $2!(n-2)!$.

$$
\begin{aligned}
\frac{(n+1)!}{3} &= 2n! \\
(n+1)! &= 6n! \\
(n+1)n! - 6n! &= 0 \\
n![(n+1)-6] &= 0
\end{aligned}
$$

Now, because $n!$ is never zero,

$$
\begin{aligned}
(n+1) - 6 &= 0 \\
n+1 &= 6 \\
n &= 5.
\end{aligned}
$$

□

Solution to Question 25:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

□

Solution to Question 26: Using the binomial theorem,

$$
\begin{aligned}
(x-2y)^3 &= (x+(-2y))^3 \\
&= \sum_{k=0}^{3} \binom{3}{k} x^{3-k}(-2y)^k.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(x-2y)^3 &= \binom{3}{0} x^3(-2y)^0 + \binom{3}{1} x^2(-2y)^1 + \binom{3}{2} x(-2y)^2 + \binom{3}{3} x^0(-2y)^3 \\
&= 1(x^3)(1) + 3(x^2)(-2y) + 3(x)(4y^2) + 1(1)(-8y^3) \\
&= x^3 - 6x^2y + 12xy^2 - 8y^3.
\end{aligned}
$$

□

Solution to Question 27: Using the binomial theorem,

$$\begin{aligned}(x-2y)^{10} &= (x+(-2y))^{10} \\ &= \sum_{k=0}^{10} \binom{10}{k} x^{10-k}(-2y)\end{aligned}$$

Because we start with $k = 0$, the 11^{th} term has $k = 10$. Thus, the 11^{th} term is

$$\begin{aligned}\binom{10}{10} x^{10-10}(-2y)^{10} &= 2x^0(-2y)^{10} \\ &= (1)(1)(1024y^{10}) \\ &= 1024y^{10}.\end{aligned}$$

□

Solution to Question 28: Using the binomial theorem,

$$\sum_{k=0}^{4} \binom{4}{k} x^{4-k}3^k = (x+3)^4.$$

Thus, the equation becomes

$$\begin{aligned}\sum_{k=0}^{4} \binom{4}{k} x^{4-k}3^k &= 1 \\ (x+3)^4 &= 1.\end{aligned}$$

Therefore,

$$\begin{aligned}x+3 &= \pm\sqrt[4]{1} \\ x+3 &= \pm 1 \\ x &= -3 \pm 1.\end{aligned}$$

Therefore, the solutions are $x = -4$ and $x = -2$. □

Solution to Question 29: Note that

$$\begin{aligned}&\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}+\binom{7}{4}+\binom{7}{5}+\binom{7}{6}+\binom{7}{7} \\ &= \binom{7}{0}1^71^0+\binom{7}{1}1^61^1+\binom{7}{2}1^51^2+\binom{7}{3}1^41^3+\binom{7}{4}1^31^4+\binom{7}{5}1^21^5+\binom{7}{6}1^11^6+\binom{7}{7}1^01^7. \\ &= \sum_{k=0}^{7}\binom{7}{k}1^{7-k}1^k \\ &= (1+1)^7 \\ &= 2^7 \\ &= 128\end{aligned}$$

□

Solutions to Exercises

Exercise 1. *Proof*: Because

$$1 = \frac{1(1+1)}{2},$$

the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{4}$$

is true for $n = 1$. Next, assume that (4) is true for $n = k$. That is, assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{5}$$

Now,

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

Thus, the statement (4) is true for $n = k + 1$.

Therefore, statement (4) is true for all $n = 1, 2, 3, \ldots$. Exercise 1