College of the Redwoods
Mathematics Department
Math 30 — College Algebra

Exam #3
College Algebra

David Arnold

Last Revision Date: April 17, 2001 Version 1.00

Multiple Choice Questions

Directions: *In each of the following exercises, select the "best" answer and darken the corresponding oval on your scantron sheet.*

1. If $f(x) = 2x^3 + 5$, which of the following best describes the inverse of the function f?

(a) $\dfrac{1}{2x^3+5}$ (b) $5x^3 - 2$ (c) $\sqrt[3]{2x+5}$

(d) $\sqrt[3]{5-2x}$ (e) $\sqrt[3]{\dfrac{x-5}{2}}$

2. If

$$f(x) = \frac{x}{x+1} \qquad \text{and} \qquad g(x) = \frac{2}{x},$$

find $(f \circ g)(x)$.

(a) $\dfrac{2}{2+x}$ (b) $\dfrac{2}{x+1}$ (c) $\dfrac{x+1}{2}$

(d) $\dfrac{x+2}{x}$ (e) $\dfrac{2x}{1+2x}$

3. The graph of $y = 4 + e^{-x}$ has an asymptote. What is its equation?

(a) $x = 4$ (b) $y = 4$ (c) $y = 3$

(d) $x = 3$ (e) none of these

4. What is the domain of $f(x) = \ln(x-6)$?

(a) $(5, +\infty)$ (b) $(-\infty, 5)$ (c) $(-\infty, 6)$

(d) $(6, +\infty)$ (e) $[6, +\infty)$

5. On the day of her birth, Elizabeth's grandmother opened a mutual fund paying 6% compounded semi-annually. Her goal is to have $40,000 available in an account on Elizabeth's 18th birthday. How much should the initial investment be to meet this goal?

(a) $11,476.40 (b) $13,801.30 (c) $15,175.40

(d) $16,753.20 (e) $18,102.54

6. A fish population obeys the logistic model of growth

$$P = \frac{5000}{20 + 30e^{-0.10t}}.$$

What is the eventual size of the population?

(a) 200 (b) 225 (c) 250

(d) 1000 (e) none of these

7. $e^{2\ln 3 + \ln 4}$ equals

(a) 24 (b) 14 (c) 36

(d) 48 (e) none of these

8. Given $\log_a 12 = 1.2442$ and $\log_a 4 = 0.2246$, find $\log_a 3$.

(a) 1.0196 (b) 5.5396 (c) 2.4442

(d) 3.6684 (e) none of these

9. Use the change of base rule to simplify

$$(\log_b a)(\log_a b).$$

(a) ab (b) $\dfrac{a}{b}$ (c) $\dfrac{b}{a}$

(d) 1 (e) none of these

10. Suppose that interest is compounded continuously at a yearly rate of 5%. How long will it take an initial investment to double in value?

(a) 11.6 yr (b) 13.9 yr (c) 12.8 yr

(d) 14.2 yr (e) none of these

Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Consider the function defined by $f(x) = x^2 + 9$, but restricted to the domain $x \leq 0$.

(a) Sketch the graph of f on a sheet of graph paper. On the same coordinate system as the graph of f, sketch the graph of f^{-1} and the line $y = x$.

(b) Use a method of your choice to find the equation of f^{-1}. Make sure that your result agrees with the work in part (a).

Exercise 2. Strontium-90 (Sr^{90}) has a half-life of 21.8 years.

(a) Use the half-life and the equation

$$N = N_0 e^{-kt}$$

to determine the *decay constant* k for Strontium-90.

(b) Using a purely analytic method (avoiding the `CALC` menu on your calculator), how long will it take a sample of 1000 milligrams to decay to 200 milligrams?

Exercise 3. Use an analytic method to solve each of the following equations for x (avoiding, again, the `CALC` menu on your calculator).

(a) $3^{x-5} = 2^{2x-3}$

(b) $\log_6 x + \log_6(x+5) = 2$

Exercise 4. Consider again the fish population that grows according to the logistic model

$$P = \frac{5000}{20 + 30e^{-t/10}},$$

where P is the number of fish and t is measured in months.

(a) Use your calculator to sketch the graph of the fish population P versus time t. Copy the result onto your examination paper. Include the `WINDOW` parameters on your sketch (`Xmin`, `Xmax`, `Ymin`, `Ymax`).

(b) Use the `CALC` menu and its `intersect` utility to find when the population first reaches 200. Include these results on the graph drawn in part (a).

(c) Use the equation $P = 5000/(20 + 30e^{-t/10})$ to find the *exact* time when the fish population first reaches 200. Show all of your analytical work and do not use your calculator to find this solution (although it is permissable to use you calculator to check that your solution agrees with the result found in part (b)). Exact answer only! No decimals allowed in the solution.

Solutions to Multiple Choice Questions

Solution to Question 1: Inevitably, evaluation of the function defined by $f(x) = 2x^3 + 5$ proceeds as follows.

1. Cube.
2. Multiply by 2.
3. Add 5.

To craft the inverse, we must "undo" each of the above operations in inverse order. That is, we proceed as follows.

1. Subtract 5.
2. Divide by 2.
3. Take the cube root.

Thus,

$$f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}.$$

Alternatively, you can find the inverse by switching x and y, then solving for y. Thus, we can write $f(x) = 2x^3 + 5$ as

$$y = 2x^3 + 5.$$

Switch x and y.

$$x = 2y^3 + 5$$

Solve for y.

$$\begin{aligned} x - 5 &= 2y^3 \\ y^3 &= \frac{x-5}{2} \\ y &= \sqrt[3]{\frac{x-5}{2}} \end{aligned}$$

Thus,

$$f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}.$$

□

Solution to Question 2: This is composition of functions, requiring that we first evaluate $g(x)$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{2}{x}\right) \end{aligned}$$

We must now replace each occurrence of x in $f(x) = x/(x+1)$ by $2/x$.

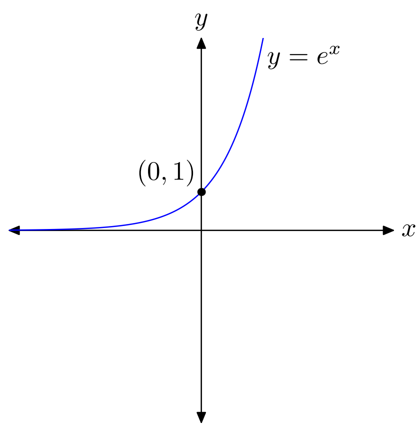
$$(f \circ g)(x) = \frac{\dfrac{2}{x}}{\dfrac{2}{x} + 1}$$

Simplify, multiplying both numerator and denominator by x.

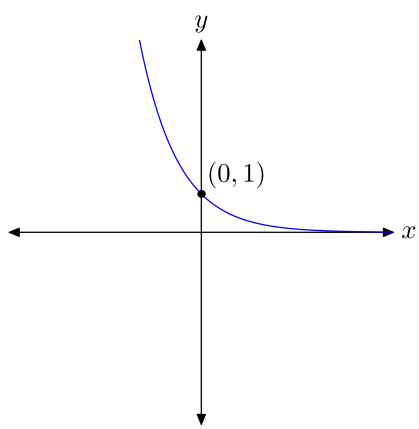
$$(f \circ g)(x) = \frac{\left(\dfrac{2}{x}\right) x}{\left(\dfrac{2}{x} + 1\right) x} = \frac{2}{2 + x}$$

□

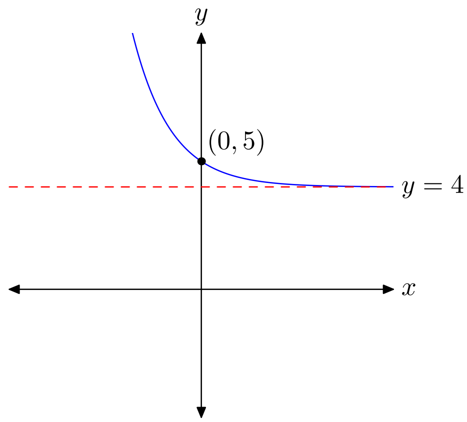
Solution to Question 3: Because $x \approx 2.7$, the graph of $y = e^x$ is increasing.



Note that the graph approaches the x-axis asymptotically as $x \to -\infty$. The graph of $y = e^{-x}$ is a reflection of the graph of $y = e^x$ across the y-axis.

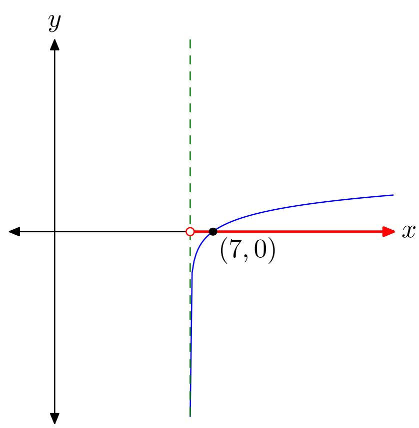


To draw the graph of $y = e^{-x} + 4$, shift the graph of $y = e^{-x}$ upward 4 units.



Thus, the graph has a horizontal asymptote with equation $y = 4$. □

Solution to Question 4: Shift the graph of $y = \ln x$ to the right 6 units to draw the graph of $f(x) = \ln(x - 6)$.



Project the points of the graph onto the x-axis to find the domain of f. Thus, the domain is $D = (6, +\infty)$. Alternatively, you cannot take the logarithm of zero or a negative number. Hence, in the expression $\ln(x-6)$, the $x - 6$ must be positive.

$$\begin{aligned} x - 6 &> 0 \\ x &> 6 \end{aligned}$$

Hence, the domain of $f(x) = \ln(x - 6)$ is $D = \{x : x > 6\}$. □

Solution to Question 5: Use the discrete compound interest formula,

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where

$$\begin{aligned} A &= \text{balance at } t \text{ years}, \\ P &= \text{principal (initial investment)}, \\ r &= \text{yearly interest rate}, \\ n &= \text{number of compounding periods}, \\ t &= \text{number of years of investment}. \end{aligned}$$

In this case, $A = 40,000$, $r = 0.06$, $n = 2$, and $t = 18$. Hence,

$$40,000 = P\left[1 + \frac{0.06}{2}\right]^{(2)(18)}.$$

Solve for P.

$$P = \frac{40,000}{\left[1 + \frac{0.06}{2}\right]^{(2)(18)}}$$

Use a calculator to approximate P.

$$P \approx 13,801.30$$

□

Solution to Question 6: Because of the negative rate, the term $e^{-0.10t}$ approaches zero as $t \to +\infty$. Hence,

$$\begin{aligned}\lim_{t\to\infty} P &= \lim_{t\to\infty} \frac{5000}{20 + 10e^{-0.10t}} \\ &= \frac{5000}{20 + 0} \\ &= 250\end{aligned}$$

Therefore, the eventual population is $P = 250$ fish. □

Solution to Question 7: First, we use $c \ln x = \ln x^c$ to write

$$\begin{aligned}e^{2\ln 3 + \ln 4} &= e^{\ln 3^2 + \ln 4} \\ &= e^{\ln 9 + \ln 4}.\end{aligned}$$

Next, use $\ln x + \ln y = \ln xy$ to write

$$e^{2\ln 3 + \ln 4} = e^{\ln 36}.$$

Finally, $e^{\ln x} = x$, so

$$e^{2\ln 3 + \ln 4} = 36.$$

□

Solution to Question 8: Because $\log_a x - \log_a y = \log_a(x/y)$, we can write

$$\begin{aligned}\log_a 3 &= \log_a \frac{12}{4} \\ &= \log_a 12 - \log_a 4 \\ &= 1.2442 - 0.2246 \\ &= 1.0196\end{aligned}$$

□

Solution to Question 9: The change of base rule, applied to each logarithm, allows us to use natural logarithms as a common base.

$$\begin{aligned}(\log_b a)(\log_a b) &= \frac{\ln a}{\ln b} \cdot \frac{\ln b}{\ln a} \\ &= 1\end{aligned}$$

□

Solution to Question 10: If interest is compounded continuously, then the balance in the account is given by

$$A = Pe^{rt},$$

where

$$\begin{aligned}A &= \text{current balance}, \\ P &= \text{initial investment}, \\ r &= \text{year by rate}, \\ t &= \text{time of investment in years}.\end{aligned}$$

If we start with an investment P, then the doubling time is found by letting $A = 2P$ and solving the resulting equation for t.

$$\begin{aligned} 2P &= Pe^{0.05t} \\ 2 &= e^{0.05t} \\ 0.05t &= \ln 2 \\ t &= \frac{\ln 2}{0.05} \end{aligned}$$

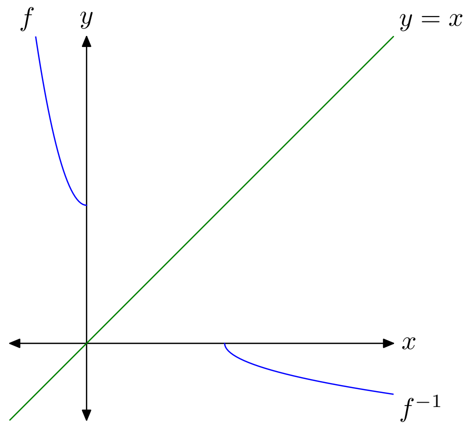
Therefore, the doubling time is

$$t \approx 13.9 \,\text{years.}$$

□

Solutions to Exercises

Exercise 1(a) The graph of $f(x) = x^2 + 9$, $x \le 0$, is a parabola that opens up and is shifted upward 9 units. However, because the domain is restricted to $x \le 0$, we only use the *left half* of the parabola as shown in the following figure.



The graph of f^{-1} is a reflection of the graph of f across the line $y = x$. □

Exercise 1(b) Let's switch x and y, then solve for y. First, $f(x) = x^2 + 9$ is equivalent to

$$y = x^2 + 9.$$

Switch x and y.

$$x = y^2 + 9$$

Solve for y.

$$y^2 = x - 9$$
$$y = \pm\sqrt{x - 9}$$

Note that there are two solutions. The graph of f^{-1} in part (a) reveals that the negative branch is the correct choice. Thus,

$$f^{-1}(x) = -\sqrt{x - 9}.$$

□

Exercise 2(a) The half life of SR^90 is 21.8 years. If we start with an initial amount N_0, then $(1/2)N_0$ remains after 21.8 years.

$$N = N_0 e^{-kt}$$
$$\frac{1}{2}N_0 = N_0 e^{-k(21.8)}$$

Divide by N_0.

$$\frac{1}{2} = e^{-21.8k}$$

Use the definition of the logarithm to write

$$-21.8k = \ln\frac{1}{2}.$$

Divide by -21.8.

$$k = \frac{\ln\frac{1}{5}}{-21.8}$$
$$k \approx 0.03180.$$

□

Exercise 2(b) Thus, our equation becomes

$$N = N_0 e^{-0.03180t}.$$

The initial amount is 1000 mg.

$$N = 1000e^{-0.03180t}$$

We want to find where the substance decays to 200 mg.

$$100 = 1000e^{-0.03180t}$$
$$\frac{200}{1000} = e^{-0.03180t}$$
$$\frac{1}{5} = e^{-0.03180t}$$

Use the definition of the logarithm to write

$$-0.03180t = \ln\frac{1}{5}$$
$$t = \frac{\ln\frac{1}{2}}{-0.03180}$$
$$t \approx 50.6\text{years}.$$

□

Exercise 3(a) Take the natural log of both sides of the equation.

$$3^{x-5} = 2^{2x-3}$$
$$\ln 3^{x-5} = \ln 2^{2x-3}$$

Bring down the exponents and distribute.

$$(x-5)\ln 3 = (2x-3)\ln 2$$
$$x\ln 3 - 5\ln 3 = 2x\ln 2 - 3\ln 2$$

Isolate the terms containing x.

$$x\ln 3 - 2x\ln 2 = 5\ln 3 - 3\ln 2$$

Factor, then divide by the coefficient of x.

$$x(\ln 3 - 2\ln 2) = 5\ln 3 - 3\ln 2$$
$$x = \frac{5\ln 3 - 3\ln 2}{\ln 3 - 2\ln 2}$$

You can use the laws of logarithms to simplify somewhat.

$$x = \frac{\ln 3^5 - \ln 2^3}{\ln 3 - \ln 2^2}$$
$$x = \frac{\ln 243 - \ln 8}{\ln 3 - \ln 4}$$
$$x = \frac{\ln\frac{243}{8}}{\ln\frac{3}{4}}$$

□

Exercise 3(b) The sum of the logs is the log of the product.

$$\log_6 x + \log_6(x+5) = 2$$
$$\log_6 x(x+5) = 2$$
$$\log_6(x^2+5x) = 2$$

Use the definition of the logarithm to write

$$x^2 + 5x = 6^2$$
$$x^2 + 5x = 36$$

Make one side zero and factor.

$$x^2 + 5x - 36 = 0$$
$$(x + 9)(x - 4) = 0$$

Thus,

$$x = -9 \qquad \text{or} \qquad x = 4.$$

We must check our answers. If $x = -9$, then

$$\log_6 x + \log_6(x + 5) = 2$$
$$\log_6(-9) + \log_6(-9 + 5) = 2$$

presents a problem because we cannot take the log of a negative number. Hence $x = -9$ is eliminated. If $x = 4$, then

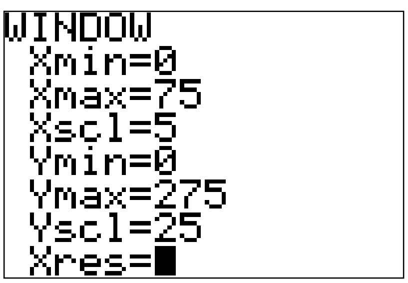
$$\log_6 x + \log_6(x + 5) = 2$$
$$\log_6 4 + \log_6(4 + 5) = 2$$
$$\log_6 4 + \log_6 9 = 2$$
$$\log_6 36 = 2.$$

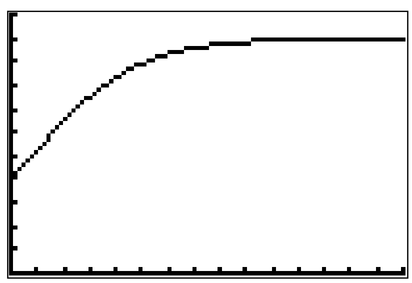
This last line is a true statement, so $x = 4$ checks and is retained as an answer. □

Exercise 4(a) Using a TI-83, the graph of

$$P = \frac{5000}{20 + 30e^{-t/10}}$$

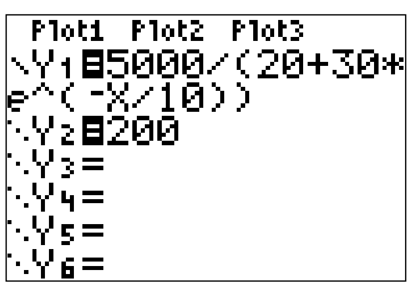
follows, including the `WINDOW` parameters used.



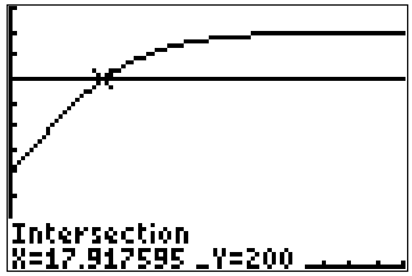


□

Exercise 4(b) Add the graph of $P = 200$ as follows



The intersect utility is used to find the time the population reaches 200.



□

Exercise 4(c) Substitute $P = 200$.

$$P = \frac{5000}{20 + 30e^{-t/10}}$$
$$200 = \frac{5000}{20 + 30e^{-t/10}}$$

Multiply both sides by the common denominator.

$$200(20 + 30e^{-t/10}) = 5000$$

Divide both sides by 200.

$$20 + 30e^{-t/10} = \frac{5000}{200}$$
$$20 + 30e^{-t/10} = 25$$

Subtract 20 from both sides.

$$30e^{-t/10} = 5$$

Divide both sides by 30.

$$e^{-t/10} = \frac{5}{30}$$
$$e^{-t/10} = \frac{1}{6}$$

Use the definition of the logarithm to write

$$-\frac{t}{10} = \ln\frac{1}{6}.$$

Multiply both sides by -10.

$$t = -10 \ln\frac{1}{6}$$
$$t \approx 17.9\,\text{months}$$

□