

# The Asteroid: Special Plane Curves

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May 15, 2006

## Abstract

In this Article we will discuss the Plane Curve known as the Asteroid. We will start with a brief history of the curve and move on to deriving the parametric equations for this curve.

## History of the Asteroid

The year is 1674, the man is Olaus (or Ole) Roemer a Dutch Astronomer following the grand tradition of Galileo. Roemer, who is most famous for first measuring the speed of light to a reasonable degree, is a good example of the well rounded Renaissance Man. He was an astronomer, but also dabbled in physics and Geometry the Renaissance language of science. During this time machinery of ropes, pulleys and gears was a “new” area, to be dabbled in by any man interested in modern science. Roemer did his share of dabbling and in 1674 postulated that if one is trying to create a machine (say for example a large clock tower), then one should use hypocycloids (wheels running within wheels) in the gears since this creates the least amount of friction between two gears, which allows them to run more efficiently. While talking about this he happened to mention using an hypocycloid with a large wheel of radius 4 times that of the smaller wheel's radius, which moves on the inside of the larger wheel. He also mentioned that the locus (shape) drawn out by this large wheel is that of a four pointed star. This is the first recorded mention of the asteroid, and thus the discovery of the asteroid is given to Roemer even though it didn't get its name for a long time.

The next and pretty much only other famous discovery regarding the asteroid was done by Daniel Bernoulli in 1725, when he came up with his famous double generation formula which states that any hypocycloidal curve can be drawn by two different hypocycloids. The idea is that if you have a circle of radius 1 as shown in Figure 1, and you place 2 smaller circles inside it, each will draw out the same shape of curve. In our case the two circles have radiuses of  $1/4$ , and  $3/4$ , the radius of the larger circle.

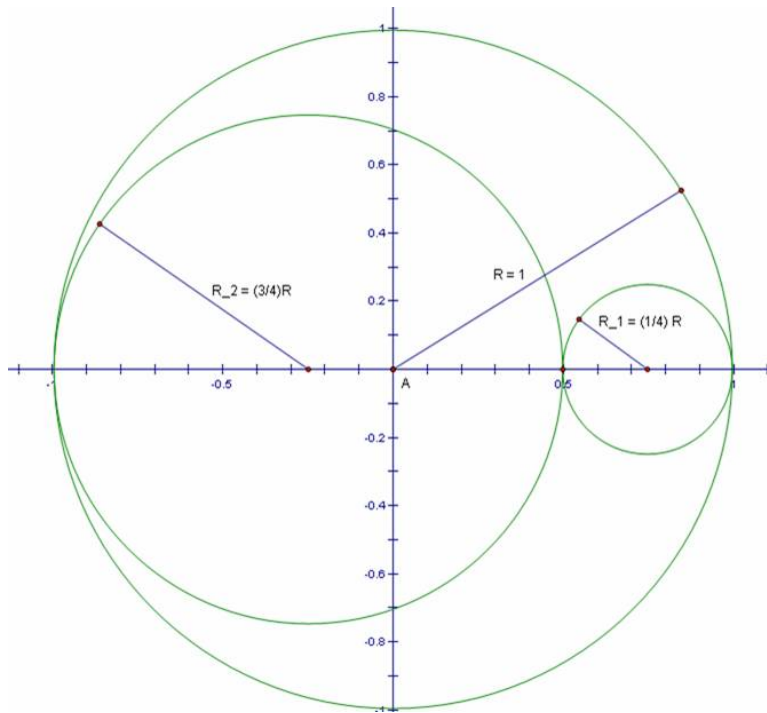


Figure 1: Double Generation of the Asteroid.

## Parameterizing the Asteroid

One advantage that we have today that the ancient Greek mathematicians didn't have is our ability to describe a curve compactly as an algebraic or trigonometric equation or set of equations. What would have taken Euclid 4 pages to describe (for example the parabola) can today, be universally understood by four simple symbols, namely  $y = x^2$ . Thus it is in our best interest to, whenever possible, derive a set of equations which describe a given curve. We will now derive the parametric equations for the asteroid. We start by looking at Figure 2, in this case, like in Roemer's original idea, we have a small circle of radius  $a$ , inside a large circle of radius  $4a$ . To get our equations we first notice that as the small circle moves around the large circle, its center is always at a distance  $3a$  from the origin. This tells that as the small circle moves it traces out another circle of radius  $3a$  (See Figure 3). Thus we note, that the curve is actually created by the combined movement of the two circles. So, we can simplify our task by finding the parametric equations of the two circles and adding them together to get the equations of the Asteroid. Since the parametric equations of a circle are  $x = r \cos(\theta)$ , and  $y = r \sin(\theta)$ , and here  $r = 3a$ , we see that the parametric equations of the larger, dotted circle is  $x = 3a \cos(\theta)$ , and  $y = 3a \sin(\theta)$ .

Now, for the smaller circle (see Figure 4). This circle has radius  $a$ , so it must have

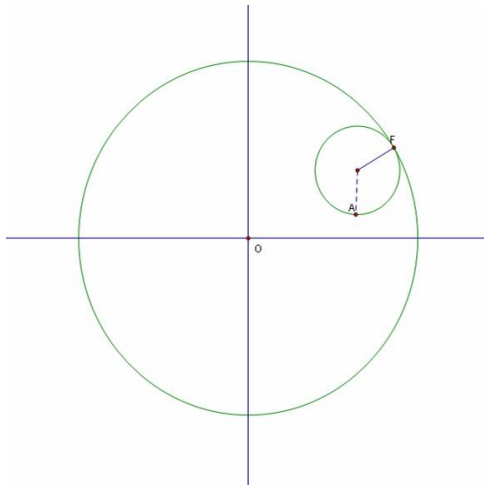


Figure 2: Initial Setup.

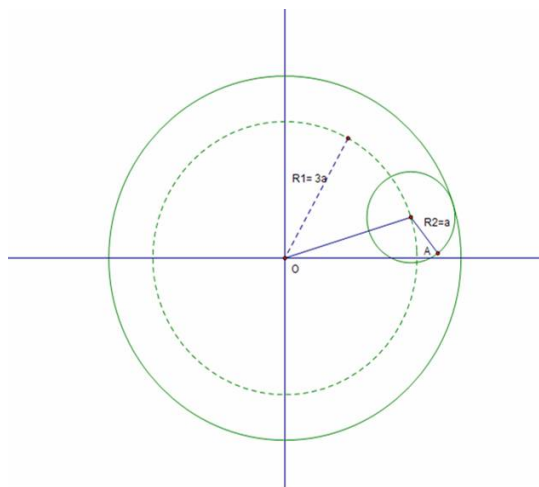


Figure 3: Large Circle Parametrization

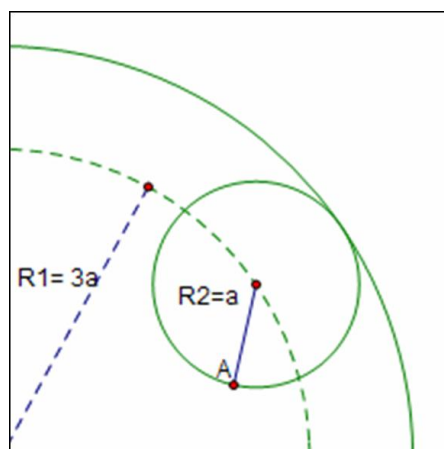


Figure 4: Smaller Circle Parametrization.

parametric equations of,  $x = a \cos(\alpha)$ , and  $y = a \sin(\alpha)$ . Finally we can add the 2 sets of parametric equations to get that the equations of the asteroid are:

$$x = 3a \cos(\theta) + a \cos(\alpha) \tag{1}$$

$$y = 3a \sin(\theta) + a \sin(\alpha) \tag{2}$$

However, we need to express  $x$  and  $y$ , in terms of one angle, (not two), to do that we need to find  $\alpha$  in terms of  $\theta$ . Here we look at the curve over the first quadrant of the graph (See Figure 5. During this time the first part goes through an angle of  $90^\circ$ , the second passes

through  $-270^\circ$ , so the relationship is  $\alpha = -3\theta$ , so our parametric equations become.

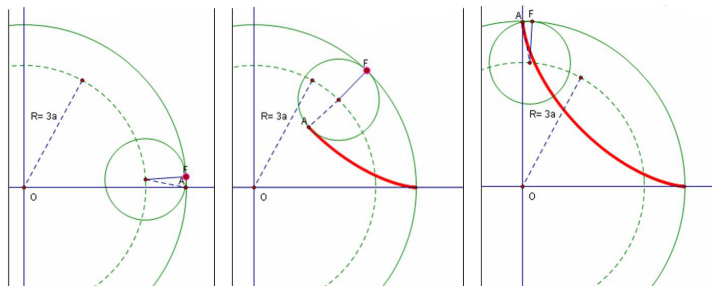


Figure 5: Relating the Angles  $\alpha$  to  $\theta$ .

$$x = 3a \cos(\theta) + a \cos(-3\theta) \quad (3)$$

$$y = 3a \sin(\theta) + a \sin(-3\theta). \quad (4)$$

Now we are going to simplify these parametric equations into more compact form. We start with our value of x, pull the negative out (cosine is an even function), and simplify.

For the X-value equation we have,

$$\begin{aligned} x &= 3a \cos(\theta) + a \cos(-3\theta) \\ &= 3a \cos(\theta) + a \cos(2\theta + \theta) \\ &= 3a \cos(\theta) + a(\cos(2\theta) \cos(\theta) - \sin(2\theta) \sin(\theta)) \\ &= 3a \cos(\theta) + a(\cos^3(\theta) - \sin^2(\theta) \cos(\theta) - 2 \sin^2(\theta) \cos(\theta)) \\ &= a \cos(\theta)(3 + \cos^2(\theta) - 3 \sin^2(\theta)) \\ &= a \cos^3(\theta) + a \cos(\theta)(3 - 3 \sin^2(\theta)) \\ &= a \cos^3(\theta) + a \cos(\theta)(3(\cos^2(\theta) + \sin^2(\theta)) - 3 \sin^2(\theta)) \\ &= a \cos^3(\theta) + a \cos(\theta)(3 \cos^2(\theta)) \\ &= a \cos^3(\theta) + 3a \cos^3(\theta) \\ &= 4a \cos^3(\theta). \end{aligned}$$

Similarity we can simplify y to  $y = 4a \sin^3(\theta)$ . Therefore our parametric equations are

$$x = 4a \cos^3(\theta) \quad (5)$$

$$y = 4a \sin^3(\theta), \quad (6)$$

In Figure 6 we can see a representation of the curve.

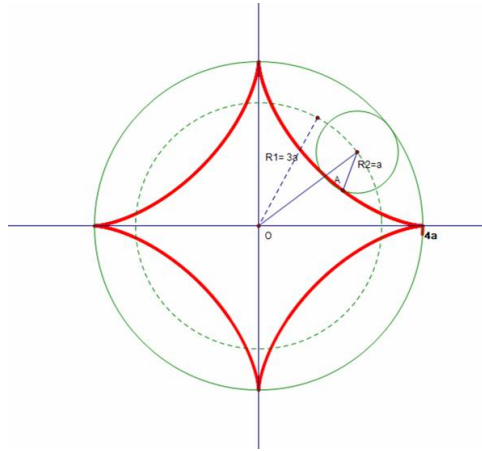


Figure 6: The Asteroid.

## Cartesian Coordinates

Once you have the parametric equations of the asteroid, getting the cartesian coordinates is no trouble at all. We simply take the cube root of  $x$  and  $y$ , then square them, and add them. We get,

$$\begin{aligned} x^{2/3} + y^{2/3} &= (4a)^{2/3} \cos^2(\theta) + (4a)^{2/3} \sin^2(\theta) \\ &= (4a)^{2/3} (\cos^2(\theta) + \sin^2(\theta)) \\ &= (4a)^{2/3} \end{aligned}$$

Now replacing radius= $4a$  with radius= $R$  we get

$$x^{2/3} + y^{2/3} = R^{2/3} \quad (7)$$

## Interesting Properties of the Asteroid

(paraphrased from Lockwood)

The evolute of the asteroid is also an asteroid, but it is rotated  $45^\circ$ , and twice the size.

The asteroid can also be drawn as the envelope of a fixed line sliding with its ends attached to the  $x$ , and  $y$  axes.

“The Asteroid is the hypocycloid formed by rolling a circle of radius  $1a$  or  $3a$  on the inside of one of radius  $4a$ ”

The Asteroid is the envelope of a family of ellipses where the sum of the major and minor axes is constant.

The Asteroid has arclength of  $6a$ , and the area inside it is  $\frac{3}{8}\pi a^2$

## References

- [1] Arnold, David. 1997, *Special Plane Curves, Assignment*, <http://online.redwoods.cc.ca.us/instruct/darnold/MULTCALC/CURVES/urves.htm>
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